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León CARVAJALES

Quantitative aspects of Anosov subgroups acting on symmetric spaces

dirigida por Rafael POTRIE y Andrés SAMBARINO

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Sr. Sébastien ÁLVAREZ	Universidad de la República	examinador
Sr. Gilles COURTOIS	CNRS-Sorbonne Université	examinador
Sra. Françoise DAL'BO	Université de Rennes 1	árbitra
Sra. Fanny KASSEL	CNRS-Institut des Hautes Études Scientifiques	examinadora
Sr. Rafael POTRIE	Universidad de la República	director
Sr. Andrés SAMBARINO	CNRS-Sorbonne Université	director
Sra. Anna WIENHARD	Universität Heidelberg	examinadora

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Resumen

El objeto de esta tesis es el estudio del problema de conteo orbital para pares simétricos pseudo-Riemannianos bajo la acción de subgrupos del tipo Anosov del grupo de Lie subyacente.

En la primera parte estudiamos este problema para el par simétrico $(\mathrm{PSO}(p,q), \mathrm{PSO}(p,q-1))$ y un subgrupo de $\mathrm{PSO}(p,q)$ de tipo proyectivamente Anosov. Miramos la órbita de una copia geodésica del espacio simétrico Riemanniano de $\mathrm{PSO}(p,q-1)$ dentro del espacio simétrico Riemanniano de $\mathrm{PSO}(p,q)$. Demostramos un comportamiento asintótico puramente exponencial, cuando t tiende a infinito, para el número de elementos en esta órbita que se encuentran a distancia menor que t de la copia geodésica original. Interpretamos este resultado como el comportamiento asintótico del número de segmentos geodésicos de tipo espacio (en el espacio hiperbólico pseudo-Riemanniano) de longitud máxima t en la órbita de un punto base. Probamos resultados análogos para otras funciones de conteo.

A continuación miramos el par $(\mathrm{PSL}_d(\mathbb{R}), \mathrm{PSO}(p,d-p))$ y un subgrupo Borel-Anosov de $\mathrm{PSL}_d(\mathbb{R})$. Presentamos contribuciones hacia la comprensión del comportamiento asintótico de la función de conteo asociada a una copia geodésica del espacio simétrico Riemanniano de $\mathrm{PSO}(p,d-p)$ en el espacio simétrico Riemanniano de $\mathrm{PSL}_d(\mathbb{R})$.

Palabras clave: *espacios simétricos, representaciones de Anosov, problemas de conteo.*

Résumé

L'objet de cette thèse est l'étude du problème de comptage orbitale pour des couples symétriques pseudo-Riemanniens sous l'action des sous-groupes de type Anosov du groupe de Lie sous-jacent.

Premièrement nous étudions ce problème pour le couple symétrique $(\mathrm{PSO}(p,q), \mathrm{PSO}(p,q-1))$ et un sous-groupe de $\mathrm{PSO}(p,q)$ de type projectivement Anosov . Nous regardons l'orbite d'une copie géodésique de l'espace symétrique Riemannien de $\mathrm{PSO}(p,q-1)$ dans l'espace symétrique Riemannien de $\mathrm{PSO}(p,q)$. Nous prouvons un comportement asymptotique purement exponentiel, lorsque t tend vers l'infini, pour le nombre d'éléments dans cette orbite qui sont à distance plus petit que t de la copie géodésique originale. Nous interprétons ce résultat comme le comportement asymptotique du nombre de segments géodésiques de type espace (dans l'espace hyperbolique pseudo-Riemannien) de longueur maximale t dans l'orbite d'un point base. Nous prouvons des résultats analogues pour d'autres fonctions de comptage.

Ensuite nous regardons le couple symétrique $(\mathrm{PSL}_d(\mathbb{R}), \mathrm{PSO}(p,d-p))$ et un sous-groupe Borel-Anosov de $\mathrm{PSL}_d(\mathbb{R})$. Nous présentons des contributions vers la compréhension du comportement asymptotique de la fonction de comptage associée à une copie géodésique de l'espace symétrique Riemannien de $\mathrm{PSO}(p,d-p)$ dans l'espace symétrique Riemannien de $\mathrm{PSL}_d(\mathbb{R})$.

Mots-clés: *espaces symétriques, représentations d'Anosov, problèmes de comptage.*

Abstract

This thesis addresses the study of the orbital counting problem for pseudo-Riemannian symmetric pairs under the action of Anosov subgroups of the underlying Lie group.

In the first part we study this problem for the pair $(\mathrm{PSO}(p,q), \mathrm{PSO}(p,q-1))$ and a projective Anosov subgroup of $\mathrm{PSO}(p,q)$. We look at the orbit of a geodesic copy of the Riemannian symmetric space of $\mathrm{PSO}(p,q-1)$ inside the Riemannian symmetric space of $\mathrm{PSO}(p,q)$. We show a purely exponential asymptotic behavior, as t goes to infinity, for the number of elements in this orbit which are at distance at most t from the original geodesic copy. We then interpret this result as the asymptotic behavior of the amount of space-like geodesic segments (in the pseudo-Riemannian hyperbolic space) of maximum length t in the orbit of a basepoint. We prove analogue results for other related counting functions.

In the second part we look at the pair $(\mathrm{PSL}_d(\mathbb{R}), \mathrm{PSO}(p,d-p))$ and a Borel-Anosov subgroup of $\mathrm{PSL}_d(\mathbb{R})$, presenting contributions towards the understanding of the asymptotic behavior of the counting function associated to a geodesic copy of the Riemannian symmetric space of $\mathrm{PSO}(p,d-p)$ inside the Riemannian symmetric space of $\mathrm{PSL}_d(\mathbb{R})$.

Key words: *symmetric spaces, Anosov representations, counting problems.*

Contents

Introduction

A *symmetric pair* is a pair (G, H) where G is a semisimple Lie group with finite center and no compact factors and H is a union of connected components of the subgroup

$$\mathrm{Fix}(\sigma) := \{g \in \mathrm{G} : \quad \sigma(g) = g\},$$

for some involutive automorphism σ of G. The associated *symmetric space* is

$$\mathsf{X} := \mathrm{G}/\mathrm{H}$$

and carries a non degenerate G-invariant metric coming from the Killing form of G (see Chapter 1). When H is a maximal compact subgroup of G this metric is Riemannian, the associated symmetric space is called the *Riemannian symmetric space* of G and it is denoted by X_{G}. When H is non compact, the pair (G, H) is called a *pseudo-Riemannian symmetric pair*. For instance, the hyperbolic plane H^2 is the Riemannian symmetric space of $\mathrm{PSL}_2(\mathbb{R})$, and the space of geodesics of H^2 is a pseudo-Riemannian symmetric space associated to $\mathrm{PSL}_2(\mathbb{R})$.

Let (G, H) be a pseudo-Riemannian symmetric pair. To a basepoint o in X one can associate a totally geodesic submanifold S^o of X_{G}, which is a copy of the Riemannian symmetric space of the stabilizer $\mathrm{H}^o \cong \mathrm{H}$ of o (see Section 1.3). The general problems that we address in this thesis are the following.

Problem A. Let (G, H) be a pseudo-Riemannian symmetric pair and Ξ be a discrete subgroup of G. Describe the set of points $o \in \mathsf{X}$ for which the orbital counting function

$$t \mapsto \#\{g \in \Xi : \quad d_{\mathsf{X}_{\mathrm{G}}}(\mathsf{S}^o, g \cdot \mathsf{S}^o) \leq t\}$$

is finite for every t and study its asymptotic behaviour as $t \to \infty$. Provide geometric interpretations for this orbital counting function in the context of the pseudo-Riemannian symmetric space X.

◇

Problem B. Study analogous questions for the function

$$t \mapsto \#\{g \in \Xi : \quad d_{\mathsf{X}_{\mathrm{G}}}(\tau, g \cdot \mathsf{S}^o) \leq t\},$$

where τ is a given point in X_{G}.

◇

In the previous formulations $d_{\mathsf{X}_{\mathrm{G}}}(\cdot, \cdot)$ denotes the distance on X_{G} coming from the G-invariant Riemannian structure and, for closed subsets A and B of X_{G}, we let

$$d_{\mathsf{X}_{\mathrm{G}}}(A, B) := \inf\{d_{\mathsf{X}_{\mathrm{G}}}(a, b) : \quad a \in A, b \in B\}.$$

More classically, the orbital counting problem concerns the study of the asymptotic behaviour of the function

$$t \mapsto \#\{g \in \Xi : \quad d_{\mathsf{X}}(o, g \cdot o) \leq t\} \tag{0.0.1}$$

as $t \to \infty$, where Ξ is a discrete group of isometries of a given proper non compact metric space X, and o is a basepoint in X. This problem has a long history and has been studied in many cases, by authors among who we find notably Gauss, Huber, Patterson and Margulis. Of course, this list is highly incomplete (we refer the reader to Babillot's survey [1] for a more complete picture). Let us mention here that the asymptotic behaviour of the function (0.0.1) has been studied when X coincides with the Riemannian symmetric space X_{G} of a semisimple Lie group G. Indeed, when $\Xi < \mathrm{G}$ is a lattice, it has been treated by Duke-Rudnick-Sarnak [19] and more generally by Eskin-McMullen [21][1]. In the non lattice case one also finds the work of Quint [57] and Sambarino [61], which deal with Δ-Anosov subgroups of G. Sambarino's approach is inspired by Roblin's method [58].

0.1 Anosov representations

Fix a semisimple Lie group G with finite center and no compact factors. In this thesis, the discrete subgroup Ξ of G that we consider will always be the image of a non elementary word hyperbolic group Γ under an *Anosov representation* $\rho : \Gamma \to \mathrm{G}$. In order to formally state our results we need to recall this notion and some of its main features. The reader familiarized with this concept can go directly to Section 0.2, where the main contributions of this thesis are properly stated.

Anosov representations are (a stable class of) faithful and discrete representations from word hyperbolic groups into semisimple Lie groups that

[1] These works also deal with pseudo-Riemannian symmetric spaces, we will come back on this point latter.

share many geometrical and dynamical features with holonomies of convex co-compact hyperbolic manifolds. They were introduced by Labourie [37] and further extended to arbitrary word hyperbolic groups by Guichard-Wienhard [26]. Since Labourie's work, Anosov representations have been object of intensive research in the field of geometric structures on manifolds and their deformation spaces as they provide a unified framework to deal with large classes of discrete subgroups of semisimple Lie groups coming from different type of constructions (see for instance the surveys of Kassel [32], Pozzetti [53] or Wienhard [64] and references therein for an account on the current state of art on the subject).

Let us recall now the definition of Anosov representations in the special case in which the target group is $\mathrm{PSL}_d(\mathbb{R})$. The definition that we present here is not Labourie's (and Guichard-Wienhard's) original definition but an equivalent definition given by Guichard-Guéritaud-Kassel-Wienhard [25] and Kapovich-Leeb-Porti [29] (see also Bochi-Potrie-Sambarino [7]). Let τ be an inner product of $\mathbb{R}^d$ and, for $g \in \mathrm{PSL}_d(\mathbb{R})$, let

$$a_1^\tau(g) \geq \cdots \geq a_d^\tau(g)$$

be the logarithms of the τ-*singular values* of g. By definition, these are the (logarithms of the) lengths of the semi axes of the ellipsoid which is the image by g of the unit sphere (associated to τ). Let θ be a non empty subset of $\Delta := \{1, \ldots, d-1\}$. A representation

$$\rho : \Gamma \to \mathrm{PSL}_d(\mathbb{R})$$

is said to be θ-*Anosov* if there exist strictly positive constants c and c' such that

$$a_j^\tau(\rho\gamma) - a_{j+1}^\tau(\rho\gamma) \geq c|\gamma|_\Gamma - c' \tag{0.1.1}$$

holds for any $\gamma \in \Gamma$ and every $j \in \theta$. Here $|\cdot|_\Gamma$ denotes the word length of Γ (with respect to some prefixed finite symmetric generating set). Note that condition (0.1.1) can be asked to hold for a representation of any finitely generated group. In fact, this condition implies that the group Γ must be word hyperbolic (see Kapovich-Leeb-Porti [31, Theorem 1.4] and also Bochi-Potrie-Sambarino [7, Section 3]).

Let $\partial_\infty\Gamma$ be the Gromov boundary of Γ. A central feature of θ-Anosov representations (see [7, 25, 29]) is that they admit a continuous equivariant *limit map*

$$\xi_{\rho,\theta} : \partial_\infty\Gamma \to \mathsf{F}_\theta.$$

Here F_θ denotes the θ-*flag variety* of G, that is, the space of tuples of the form

$$(\xi^{j_1} \subset \cdots \subset \xi^{j_i})$$

where $\theta = \{j_1, \ldots, j_i\}$ and, for every $k = 1, \ldots, i$ the element ξ^{j_k} is a j_k-dimensional subspace of $\mathbb{R}^d$. It can be proven further that the map $\xi_{\rho,\theta}$ is *transverse* and *dynamics preserving* (see Section 2.1 for precisions). The image of $\xi_{\rho,\theta}$ is called the *θ-limit set* of ρ.

The reader is referred to the surveys mentioned above [32, 53, 64] for a detailed discussion of examples of Anosov representations (which include notably *Schottky representations* and *Hitchin representations*). Some of these examples will be discussed in Section 2.1.

Let us remark here that the notion of Anosov representation can be extended to representations into G. In this setting, the choice of θ is substituted by the choice of a conjugacy class of parabolic subgroups of G (see Chapter 2).

0.2 Contributions of the thesis

In order to state our results we need to fix some notations and general terminology. Let (G, H) be a pseudo-Riemannian symmetric pair and o be a basepoint in X. We denote by σ^o the corresponding involution of G and note that the tangent space $T_o\mathsf{X}$ identifies with

$$\mathfrak{q}^o := \{d\sigma^o = -1\},$$

where $d\sigma^o$ denotes the differential of σ^o at the identity element of G. Let τ be a point in S^o and σ^τ be the corresponding involutive automorphism of G (it is a *Cartan involution*). Define

$$\mathfrak{p}^\tau := \{d\sigma^\tau = -1\} \cong T_\tau\mathsf{X}_{\mathrm{G}}.$$

Maximal abelian subalgebras of $\mathfrak{p}^\tau \cap \mathfrak{q}^o$ (resp. $\mathfrak{p}^\tau$) will be denoted with the symbol $\mathfrak{b}$ (resp. $\mathfrak{a}$). The dimension of $\mathfrak{b}$ (resp. $\mathfrak{a}$) is called the *rank* of the pseudo-Riemannian symmetric space X (resp. Riemannian symmetric space X_{G}). Geometrically, the submanifold

$$\exp(\mathfrak{b}) \cdot o$$

is a totally geodesic *space-like flat* in X. Here *flat* means that sectional curvature vanishes over this submanifold and *space-like* means that the restriction of the Killing form to tangent spaces of $\exp(\mathfrak{b}) \cdot o$ is positive definite (see Section 1.1 for further precisions). The norm on $\mathfrak{b}$ given by this restriction is denoted by $\|\cdot\|_{\mathfrak{b}}$.

Finally, let

$$\mathfrak{g}^{\tau o} := \{d\sigma^o d\sigma^\tau = 1\}.$$

Weyl chambers of the system $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$ (resp. $\Sigma(\mathfrak{g}, \mathfrak{a})$) will be denoted by $\mathfrak{b}^+$ (resp. $\mathfrak{a}^+$). For further precisions and more concrete descriptions of these objects in the examples that interest us see Sections 1.1 and 1.4.

0.2.1 The case $\mathrm{G} = \mathrm{PSO}(p,q)$ and $\mathrm{H} = \mathrm{PSO}(p,q-1)$

What we present here is the subject of the article C. [14].

Fix two integers $p \geq 1$ and $q \geq 2$ and denote by $\mathbb{R}^{p,q}$ the vector space $\mathbb{R}^{p+q}$ endowed with the bilinear symmetric form $\langle \cdot, \cdot \rangle_{p,q}$ defined by

$$\langle (x_1, \dots, x_{p+q}), (y_1, \dots, y_{p+q}) \rangle_{p,q} := \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i.$$

Until otherwise stated, G will denote the Lie group $\mathrm{PSO}(p,q)$ of projectivized matrices in $\mathrm{SL}_d(\mathbb{R})$ preserving the form $\langle \cdot, \cdot \rangle_{p,q}$. The first pseudo-Riemannian symmetric pair that we look at is the following

$$(\mathrm{G}, \mathrm{H}) := (\mathrm{PSO}(p,q), \mathrm{PSO}(p,q-1)),$$

where $\mathrm{H} = \mathrm{PSO}(p,q-1)$ is embedded in G as the stabilizer of a line of $\mathbb{R}^{p,q}$ which is negative for the form $\langle \cdot, \cdot \rangle_{p,q}$. The corresponding involution of G is given by conjugation by the matrix that acts as id (resp. $-$id) on that line (resp. the $\langle \cdot, \cdot \rangle_{p,q}$-orthogonal complement of that line).

The associated pseudo-Riemannian symmetric space is denoted by $\mathsf{H}^{p,q-1}$ and it is called the *pseudo-Riemannian hyperbolic space of signature* $(p, q-1)$. We think here the space $\mathsf{H}^{p,q-1}$ as the space of lines in $\mathbb{R}^{p,q}$ on which the form $\langle \cdot, \cdot \rangle_{p,q}$ is negative. The rank of $\mathsf{H}^{p,q-1}$ is equal to one (see Example 1.2.4).

A concrete description of the Riemannian symmetric space X_G as well as the submanifold S^o (for a given $o \in \mathsf{H}^{p,q-1}$) can be found in Sections 1.1 and 1.3.

Given a $\{1\}$-Anosov representation

$$\rho : \Gamma \to \mathrm{G}$$

we consider the set[2]

$$\boldsymbol{\Omega}_\rho := \{ o \in \mathsf{H}^{p,q-1} : \quad \langle o, \xi_\rho(x) \rangle_{p,q} \neq 0 \text{ for all } x \in \partial_\infty \Gamma \},$$

where ξ_ρ is the limit map of ρ in projective space. In the study of discrete groups of projective transformations, it is standard to consider sets similar to $\boldsymbol{\Omega}_\rho$ (see for instance Danciger-Guéritaud-Kassel [18, 17] and references therein). Without any further assumption the set $\boldsymbol{\Omega}_\rho$ could be empty. A large class of examples of representations ρ for which the subset $\boldsymbol{\Omega}_\rho$ is non empty is given by $\mathsf{H}^{p,q-1}$-*convex co-compact representations* introduced by Danciger-Guéritaud-Kassel [18, 17] (c.f. Example 2.1.10 and Section 4.1).

We first show the following (see Propositions 4.4.1 and 4.4.2).

[2] Here we abuse notations because o and $\xi_\rho(x)$ are points in the projective space $\mathsf{P}(\mathbb{R}^{p,q})$. Note however that the condition $\langle o, \xi_\rho(x) \rangle_{p,q} \neq 0$ does not depend on the choice of representatives of these lines in $\mathbb{R}^{p,q}$.

Proposition (C. [14]). *Let $\rho : \Gamma \to \mathrm{G}$ be a $\{1\}$-Anosov representation and fix points $o \in \mathbf{\Omega}_\rho$ and $\tau \in \mathsf{S}^o$. Then for every $t \geq 0$ one has*

$$\#\{\gamma \in \Gamma : \quad d_{\mathsf{X}_\mathrm{G}}(\tau, \rho\gamma \cdot \mathsf{S}^o) \leq t\} < \infty$$

and

$$\#\{\gamma \in \Gamma : \quad d_{\mathsf{X}_\mathrm{G}}(\mathsf{S}^o, \rho\gamma \cdot \mathsf{S}^o) \leq t\} < \infty.$$

Our main contributions in the present framework are Theorems A and B. The notation $f(t) \sim g(t)$ stands for

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

Theorem A (C. [14]). *Let $\rho : \Gamma \to \mathrm{G}$ be a $\{1\}$-Anosov representation and $o \in \mathbf{\Omega}_\rho$. There exist constants $h_\rho > 0$ and $\mathtt{m}_{\rho,o} > 0$ such that*

$$\#\{\gamma \in \Gamma : \quad d_{\mathsf{X}_\mathrm{G}}(\mathsf{S}^o, \rho\gamma \cdot \mathsf{S}^o) \leq t\} \sim \frac{e^{h_\rho t}}{\mathtt{m}_{\rho,o}}.$$

Theorem B (C. [14]). *Let $\rho : \Gamma \to \mathrm{G}$ be a $\{1\}$-Anosov representation, a point $o \in \mathbf{\Omega}_\rho$ and $\tau \in \mathsf{S}^o$. There exist constants $h_\rho > 0$ and $\mathtt{m}_{\rho,o,\tau}$ such that*

$$\#\{\gamma \in \Gamma : \quad d_{\mathsf{X}_\mathrm{G}}(\tau, \rho\gamma \cdot \mathsf{S}^o) \leq t\} \sim \frac{e^{h_\rho t}}{\mathtt{m}_{\rho,o,\tau}}.$$

The constant h_ρ is the same in both Theorems A and B and it is independent on the choice of o in $\mathbf{\Omega}_\rho$ (and τ in S^o). It coincides with the topological entropy of the *geodesic flow* ϕ^ρ of ρ, introduced by Bridgeman-Canary-Labourie-Sambarino [12], and can be computed as

$$h_\rho = \limsup_{t\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \quad \lambda_1(\rho\gamma) \leq t\}}{t}.$$

Here $[\gamma]$ denotes the conjugacy class of γ and $\lambda_1(\rho\gamma)$ denotes the logarithm of the spectral radius of $\rho\gamma$. The constants $\mathtt{m}_{\rho,o}$ and $\mathtt{m}_{\rho,o,\tau}$ are related to the total mass of specific measures in the Bowen-Margulis measure class of ϕ^ρ (recall that the Bowen-Margulis measure class is the homothety class of measures maximizing entropy of ϕ^ρ).

Let us mention some previous work that has some intersection with Theorems A and B (the reader is referred to Parkkonen-Paulin's survey [47] for a more complete picture):

- When p equals one, the space X_{G} coincides with the hyperbolic space H^q of dimension q. In that case Theorem A is covered by a result of Parkkonen-Paulin [48]. The results of [48] are valid also in some situations of variable strictly negative curvature and in some of these situations the authors obtain estimates on the error terms for their counting results (see [48] for precisions). On the other hand, Theorem B is also covered by [48] when p equals one. In this case, the results of [48] generalize work of Oh-Shah [46], Lee-Oh [39] and Mohammadi-Oh [45].

- When p is strictly bigger than one the space X_{G} is *higher rank*, i.e. it has non positive curvature but contains isometric copies of $\mathbb{R}^l$ for some $l \geq 2$. We remark here that Eskin-McMullen's work [21] deals with pseudo-Riemannian symmetric pairs as well and that they obtain counting results similar to Theorem B in some situations that include higher rank Lie groups. We emphasize however that, in contrast with [21], the discrete subgroup Ξ we look at here is not a lattice.

Since the work of Margulis [41], in order to obtain a counting result one usually studies the ergodic properties of a well chosen dynamical system. All the works quoted above use this approach and in particular rely on the mixing property of some appropriate measure invariant under some appropriate flow. Here we follow the approach by Sambarino [60] and construct a dynamical system on a compact space that contains the required geometric information. In order to do that is useful to look at the space $\mathsf{H}^{p,q-1}$. Further, in the process of this construction we obtain geometric interpretations for Theorems A and B in this pseudo-Riemannian setting.

Let $o, o' \in \mathsf{H}^{p,q-1}$ be two points joined by a space-like geodesic and let $\mathtt{l}_{o,o'}$ be the length of this geodesic segment (see Subsection 3.1.1). We denote by $\mathscr{C}_o^>$ the set of points of $\mathsf{H}^{p,q-1}$ that can be joined to o by a space-like geodesic and we define

$$\mathscr{C}_{o,\mathrm{G}}^> := \{g \in \mathrm{G} : \quad g \cdot o \in \mathscr{C}_o^>\}.$$

The following gives a Lie theoretic description of the subset $\mathscr{C}_{o,\mathrm{G}}^>$ which can be thought as a "Cartan Decomposition" adapted to our setting (see Proposition 3.2.1).

Proposition (C. [14])**.** *Let $o \in \mathsf{H}^{p,q-1}$, $\tau \in \mathsf{S}^o$ and $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ be a maximal subalgebra. Fix a Weyl chamber $\mathfrak{b}^+$. Given $g \in \mathscr{C}_{o,\mathrm{G}}^>$ there exists $h, h' \in \mathrm{H}^o$ and a unique $X \in \mathfrak{b}^+$ such that*

$$g = h \exp(X) h'.$$

We then introduce a generalized Cartan projection

$$b^o : \mathscr{C}^{>}_{o,\mathrm{G}} \to \mathfrak{b}^+$$

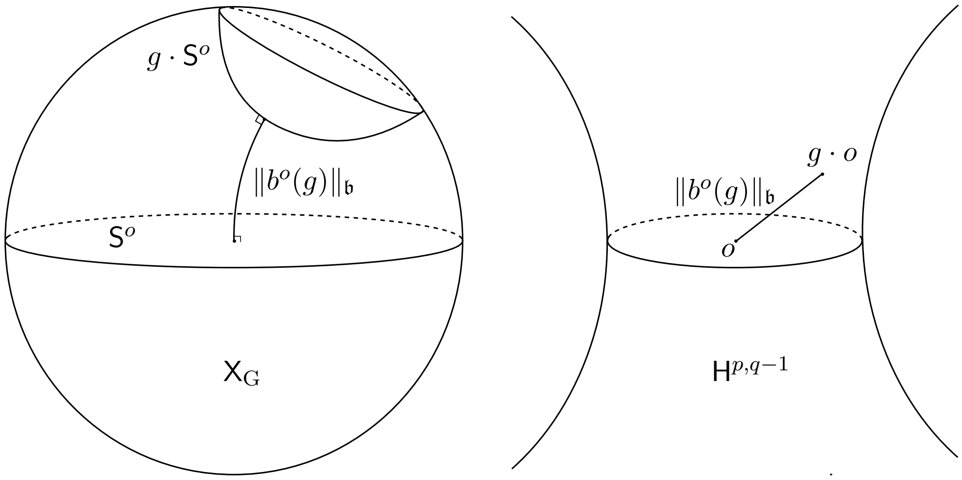
by means of the equality

$$g = h \exp(b^o(g))h'$$

for some $h, h' \in \mathrm{H}^o$. We deduce the following (see Proposition 3.2.2).

Proposition (C. [14])**.** *Let $o \in \mathsf{H}^{p,q-1}$ and $g \in \mathscr{C}^{>}_{o,\mathrm{G}}$. Then*

$$d_{\mathsf{X}_{\mathrm{G}}}(\mathsf{S}^o, g \cdot \mathsf{S}^o) = \|b^o(g)\|_{\mathfrak{b}} = \mathtt{l}_{o,g\cdot o}.$$



In Corollary 4.2.2 we prove that given a $\{1\}$-Anosov representation $\rho : \Gamma \to \mathrm{G}$ and a basepoint o in $\mathbf{\Omega}_\rho$, then apart from possibly finitely many exceptions γ in Γ one has $\rho\gamma \in \mathscr{C}^{>}_{o,\mathrm{G}}$. Theorem A now becomes

$$\#\{\gamma \in \Gamma : \quad \rho\gamma \in \mathscr{C}^{>}_{o,\mathrm{G}} \text{ and } \mathtt{l}_{o,\rho\gamma\cdot o} \leq t\} \sim \frac{e^{h_\rho t}}{\mathtt{m}_{\rho,o}}.$$

The following proposition, which is again deduced from Proposition 3.2.1, is the key ingredient in our approach because it provides a tractable way of working with the projection b^o (see Proposition 3.2.4 for a proof).

Proposition (C. [14])**.** *For every g in $\mathscr{C}^{>}_{o,\mathrm{G}}$ one has*

$$\|b^o(g)\|_{\mathfrak{b}} = \tfrac{1}{2}\lambda_1(\sigma^o(g)g^{-1}).$$

Proposition 3.2.4 allows us to apply Ledrappier's [38] and Sambarino's [60] framework to our setting and suggests the construction of some particular flow space that allows us to obtain the desired counting result (see the introduction of Part II for complementary information).

Before discussing the next symmetric pair we look at in this thesis, we make few remarks:

- Glorieux-Monclair [23] introduced an orbital counting function for $\mathsf{H}^{p,q-1}$-convex co-compact representations that differs from

$$t \mapsto \#\{\gamma \in \Gamma : \quad \rho\gamma \in \mathscr{C}^{>}_{o,\mathrm{G}} \text{ and } \daleth_{o,\rho\gamma\cdot o} \leq t\}$$

by a constant. Indeed, they define an $\mathsf{H}^{p,q-1}$*-distance*

$$d_{\mathsf{H}^{p,q-1}}(o,o') := \begin{cases} \daleth_{o,o'} & \text{if } o' \in \mathscr{C}^{>}_{o} \\ 0 & \text{otherwise} \end{cases},$$

and show that it satisfies a version of the triangle inequality in the convex hull of the limit set of ρ. This is used to prove that the exponential growth rate of the counting function

$$t \mapsto \#\{\gamma \in \Gamma : \quad d_{\mathsf{H}^{p,q-1}}(o, \rho\gamma \cdot o) \leq t\}$$

is independent on the choice of the basepoint o. The authors interpret this exponential rate as a *pseudo-Riemannian Hausdorff dimension* of the limit set of ρ, with the purpose of finding upper bounds for this number (see [23, Theorem 1.2]). A consequence of Theorem A and Proposition 3.2.2 (see Remarks 4.4.3 and 5.1.12) is that this rate coincides with the topological entropy h_ρ of ϕ^ρ.

- A counterpart of Theorem B in the space $\mathsf{H}^{p,q-1}$ is also available and the link in this case is provided by the well known *polar projection* of G. To keep this introduction in a reasonable size we refer the reader to Section 1.4 for this link, but we mention here that the relation between Theorem B and the polar projection of G addresses the problems treated by Kassel-Kobayashi in [33, Section 4]. Indeed, the authors study the orbital counting function of Theorem B for *sharp* subgroups of a real reductive symmetric space (see [33, Section 4]). Kassel-Kobayashi obtain some estimates on the growth of this function, but no precise asymptotic is established. We mention also that very recently Edwards-Lee-Oh [20] published a preprint in which they prove a counting theorem for the polar projection of a symmetric pair. Their result is therefore related to Theorem B (c.f. [20, Theorem 1.11]).

The method of [23] is based on pseudo-Riemannian geometry. As outlined earlier, our approach has Lie theoretic flavor and it is inspired by [33].

0.2.2 The case $\mathrm{G} = \mathrm{PSL}(V)$ and $\mathrm{H} = \mathrm{PSO}(p, q)$

What we present here is work still in progress.

Let V be a real vector space of dimension $d \geq 2$. In this section G will denote the Lie group $\mathrm{PSL}(V)$ of projectivized matrices in $\mathrm{SL}(V)$. Fix two integers $p \geq 1$ and $q \geq 1$ such that $d = p+q$. The second pseudo-Riemannian symmetric pair that we look at is the following

$$(\mathrm{G}, \mathrm{H}) := (\mathrm{PSL}(V), \mathrm{PSO}(p, q)),$$

where $\mathrm{H} = \mathrm{PSO}(p, q)$ is embedded in G as the stabilizer of a quadratic form on V of signature (p, q). Explicitly, the associated involution of G is given by

$$g \mapsto {}^{\star}g^{-1},$$

where ${}^{\star}\cdot$ is the adjoint operator induced by this quadratic form.

The associated pseudo-Riemannian symmetric space is denoted by $\mathsf{Q}_{p,q}$ and we think it as the space homothety classes[3] of quadratic forms on V of signature (p, q). In contrast with the previous case, the rank of $\mathsf{Q}_{p,q}$ equals $d-1$: maximal subalgebras in $\mathfrak{p}^{\tau} \cap \mathfrak{q}^{o}$ are in fact maximal in $\mathfrak{p}^{\tau}$ (see Example 1.2.6). That is, in this case we can take $\mathfrak{b} = \mathfrak{a}$ and we have the inclusion $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b}) \subset \Sigma(\mathfrak{g}, \mathfrak{a})$.

Given a Δ-Anosov representation

$$\rho : \Gamma \to \mathrm{G}$$

we define[4]

$$\mathbf{\Omega}_{\rho} := \{o \in \mathsf{Q}_{p,q} : \quad \xi_{\rho}(x) \text{ is transverse to } \xi_{\rho}(x)^{\perp_o} \text{ for all } x \in \partial_{\infty}\Gamma\},$$

where ξ_{ρ} is the limit map of ρ in the space of full flags $\mathsf{F}(V)$ of V. Another way of expressing the set $\mathbf{\Omega}_{\rho}$ is the following: a point o belongs to $\mathbf{\Omega}_{\rho}$ if and only if the limit set $\xi_{\rho}(\partial_{\infty}\Gamma)$ of ρ is contained in the union of the open orbits of the action

$$\mathrm{H}^{o} \curvearrowright \mathsf{F}(V),$$

[3] Two quadratic forms on V belong to the same homothety class if they differ by multiplication by a strictly positive real number.

[4] For a quadratic form $o \in \mathsf{Q}_{p,q}$ and a full flag $\xi = (\xi^1, \ldots, \xi^d)$, the flag $\xi^{\perp_o} = (\tilde{\xi}^1, \ldots, \tilde{\xi}^d)$ is defined by the equalities

$$\tilde{\xi}^j := (\xi^{d-j})^{\perp_o}$$

for every $j = 1, \ldots, d$. Here $\cdot^{\perp_o}$ denotes the orthogonal complement with respect to the form o. Recall that ξ is *transverse* to $\xi^{\perp_o}$ if ξ^j is linearly disjoint from $\tilde{\xi}^{d-j}$ for every $j = 1, \ldots, d$.

where H^o denotes the stabilizer of $o \in \mathsf{Q}_{p,q}$. We refer the reader to Section 6.4 for further precisions on this point. Examples of representations ρ for which the subset $\mathbf{\Omega}_\rho$ is non empty are discussed latter in this introduction (see also Section 9.1).

Counting problems in higher rank are more involved than in rank one (c.f. [57, 61]). Nevertheless, in Subsection 9.1.2 we can prove the following.

Proposition (Corollaries 9.1.2 and 9.1.3)**.** *Let $\rho : \Gamma \to \mathrm{G}$ be a Δ-Anosov representation and o be a point in $\mathbf{\Omega}_\rho$. Then for every positive t one has that*

$$\#\{\gamma \in \Gamma : \quad d_{\mathsf{X}_\mathrm{G}}(\mathsf{S}^o, \rho\gamma \cdot \mathsf{S}^o) \leq t\} \tag{0.2.1}$$

is finite. Moreover, the exponential growth rate

$$\limsup_{t\to\infty} \frac{\log \# \{\gamma \in \Gamma : \quad d_{\mathsf{X}_\mathrm{G}}(\mathsf{S}^o, \rho\gamma \cdot \mathsf{S}^o) \leq t\}}{t}$$

is positive, finite and independent on the choice of the basepoint $o \in \mathbf{\Omega}_\rho$. It coincides with

$$\delta_\rho := \limsup_{t\to\infty} \frac{\log \# \{\gamma \in \Gamma : \quad d_{\mathsf{X}_\mathrm{G}}(\tau, \rho\gamma \cdot \tau) \leq t\}}{t}$$

for any point $\tau \in \mathsf{X}_\mathrm{G}$.

Our main objective is to find an asymptotic of the counting function (0.2.1) as $t \to \infty$. We will now describe some partial results in this direction (the main one being Theorem C below). In order to do that we introduce, for a given basepoint $o \in \mathsf{Q}_{p,q}$, the subset

$$\mathscr{B}_{o,\mathrm{G}}$$

consisting on elements of G that take some o-orthogonal basis of lines of V into another o-orthogonal basis of lines of V. Here by *o-orthogonal basis of lines* of V we mean a set of d different lines in V which are pairwise orthogonal with respect to the quadratic form o. Note that when o is a (positive or negative) definite form, the set $\mathscr{B}_{o,\mathrm{G}}$ coincides with G. However, we are assuming that o is indefinite and in this case $\mathscr{B}_{o,\mathrm{G}}$ is strictly contained in G.

Geometrically, the subset $\mathscr{B}_{o,\mathrm{G}}$ coincides with the set of elements g in G for which there exists a geodesic copy of $\mathbb{R}^{d-1}$ inside X_G both orthogonal to S^o and $g \cdot \mathsf{S}^o$ (see Subsection 7.3.1). On the other hand, the Lie theoretic description of $\mathscr{B}_{o,\mathrm{G}}$ is as follows. Fix a point $\tau \in \mathsf{S}^o$ and a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$. Fix as well a Weyl chamber $\mathfrak{b}^+ \subset \mathfrak{b}$ of the system $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$. Recall that we can take $\mathfrak{b} = \mathfrak{a}$ in this case. We let W be the Weyl group of $(\mathfrak{g}, \mathfrak{a})$.

Proposition (Propositions 7.1.1 and 7.1.2)**.** *The subset* $\mathscr{B}_{o,\mathrm{G}}$ *decomposes as*

$$\mathscr{B}_{o,\mathrm{G}} = \mathrm{H}^o \mathrm{W} \exp(\mathfrak{b}^+) \mathrm{H}^o.$$

Further, the $\mathfrak{b}^+$*-coordinate in this decomposition is uniquely determined.*

The presence of the "W-coordinate" in the previous decomposition can be roughly explained by the fact that the choice of the subalgebra $\mathfrak{b}$ does not determine a space-like flat in $\mathsf{Q}_{p,q}$ but rather a disjoint union of space-like flats in $\mathsf{Q}_{p,q}$. This disjoint union is parametrized by certain subset of the Weyl group W which is in one to one correspondence with

$$(\mathrm{W} \cap \mathrm{H}^o)\backslash \mathrm{W}.$$

The reader is referred to Subsection 6.1.2 for further precisions.

The decomposition of $\mathscr{B}_{o,\mathrm{G}}$ given by the previous proposition will be called (p,q)-*Cartan decomposition*. We then introduce a (p,q)-*Cartan projection*

$$b^o : \mathscr{B}_{o,\mathrm{G}} \to \mathfrak{b}^+$$

characterized by the equation

$$g = hw \exp(b^o(g)) h'$$

for every $g \in \mathscr{B}_{o,\mathrm{G}}$, where $h, h' \in \mathrm{H}^o$ and $w \in \mathrm{W}$. As in the previous case, in Lemma 7.3.1 we show the equality

$$\|b^o(g)\|_{\mathfrak{b}} = d_{\mathsf{X}_\mathrm{G}}(\mathsf{S}^o, g \cdot \mathsf{S}^o).$$

However, the presence of the "W-coordinate" makes the geometric interpretation of the projection b^o in the pseudo-Riemannian setting less clear (see Subsection 7.3.2 for discussions).

Fix a Δ-Anosov representation $\rho : \Gamma \to \mathrm{G}$ and a basepoint $o \in \mathbf{\Omega}_\rho$. In Corollary 9.1.2 we show that apart from possibly finitely many exceptions $\gamma \in \Gamma$ one has

$$\rho\gamma \in \mathscr{B}_{o,\mathrm{G}}.$$

We briefly (and informally) outline some examples. Suppose that $d = 3$ and $p = 2 = q + 1$. In Example 9.2.2 we exhibit the following:

- If $\rho : \Gamma \to \mathrm{G}$ is a *Hitchin representation* (see Example 2.1.8) and o is a basepoint in $\mathbf{\Omega}_\rho$, then the (p,q)-Cartan decomposition of (large) elements $\rho\gamma$ always has a non trivial "W-coordinate". Geometrically, this means that if we consider a maximal geodesic flat subspace $[f] \subset \mathsf{X}_\mathrm{G}$ both orthogonal to S^o and $\rho\gamma \cdot \mathsf{S}^o$ and that intersects these submanifolds respectively in τ_1 and τ_2, then the following phenomenon occurs: if we translate the submanifold $\rho\gamma \cdot \mathsf{S}^o$ "along" the geodesic segment connecting τ_2 with τ_1 we do **not** obtain S^o but rather the submanifold S^o "shifted" by an element of the Weyl group of $[f]$.

- On the contrary, if $\rho : \Gamma \to \mathrm{G}$ is of *Barbot type* (see Example 2.1.9) and o is a basepoint in $\mathbf{\Omega}_\rho$, then the (p,q)-Cartan decomposition of (large) elements $\rho\gamma$ always has trivial "W-coordinate".

We now continue with the exposition of our results. As a next step we introduce the *asymptotic cone* $\mathscr{L}_\rho^{p,q}$ of ρ as the subset of $\mathfrak{b}^+$ consisting on all possible limits of the form

$$\frac{b^o(\rho\gamma_n)}{t_n}$$

where $t_n \to \infty$. This is the analogue of Benoist's asymptotic cone $\mathscr{L}_\rho$, which was introduced in [4] in the same way, but using the usual Cartan projection

$$a^\tau : \mathrm{G} \to \mathfrak{a}^+$$

of G instead of the projection b^o. We show the following.

Proposition (Proposition 9.1.6)**.** *Fix a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{b}^+$ of the system $\Sigma(\mathfrak{g},\mathfrak{a})$. Then there exists a subset $\mathrm{W}_{\rho,\mathfrak{a}^+}$ of the Weyl group W for which one has*

$$\mathscr{L}_\rho^{p,q} = \bigcup_{w \in \mathrm{W}_{\rho,\mathfrak{a}^+}} w \cdot \mathscr{L}_\rho.$$

The subset $\mathrm{W}_{\rho,\mathfrak{a}^+}$ is explicitly described in terms of the open orbits of the action of $\mathrm{H}^o \curvearrowright \mathsf{F}(V)$ that intersect the limit set $\xi_\rho(\partial_\infty\Gamma)$. In particular, the limit cone $\mathscr{L}_\rho^{p,q}$ is not necessarily convex (c.f. Benoist [4]). Whenever the limit set of ρ is contained in a single open orbit of the action $\mathrm{H}^o \curvearrowright \mathsf{F}(V)$, the Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{b}^+$ can be chosen in such a way that the equality

$$\mathscr{L}_\rho^{p,q} = \mathscr{L}_\rho$$

holds (see Remark 9.1.4).

The main result that we obtain in the present framework is the following.

Theorem C (Proposition 9.2.10)**.** *Let $\rho:\Gamma\to \mathrm{G}$ be a Zariski dense Δ-Anosov representation and o be a basepoint in $\mathbf{\Omega}_\rho$. Suppose further that there exists a unique open orbit of the action*

$$\mathrm{H}^o \curvearrowright \mathsf{F}(V)$$

that contains the limit set $\xi_\rho(\partial_\infty\Gamma)$. Then for every linear functional $\varphi\in\mathfrak{b}^*$ which is strictly positive in the interior of $\mathscr{L}_\rho^{p,q}$ there exist constants $h_\rho^\varphi>0$ and $\mathtt{m}_{\rho,o,\varphi}>0$ such that

$$\#\{\gamma\in\Gamma: \quad \rho\gamma\in\mathscr{B}_{o,\mathrm{G}} \text{ and } \varphi(b^o(\rho\gamma))\leq t\}\sim \frac{e^{h_\rho^\varphi t}}{\mathtt{m}_{\rho,o,\varphi}}.$$

The constant h_ρ^φ in Theorem C coincides with the φ-*entropy* of ρ, which is defined by

$$h_\rho^\varphi := \lim_{t\to\infty}\frac{\log\#\{[\gamma]\in[\Gamma]: \quad \varphi(\lambda(\rho\gamma))\leq t\}}{t}.$$

Here $\lambda(\cdot)$ denotes the Jordan projection of G. In other words, h_ρ^φ is the topological entropy of certain reparametrization of the geodesic flow ϕ^ρ of ρ (see Subsection 2.3.3). On the other hand, the constant $\mathtt{m}_{\rho,o,\varphi}$ is related with the total mass of a specific measure in the Bowen-Margulis measure class of this reparametrization. The Zariski density assumption in Theorem C is needed to ensure that this reparametrization is *topologically weakly-mixing*. This is a necessary input to obtain the desired counting result (see Appendix A for further precisions).

We find the following bound for our main problem (recall that δ_ρ is the exponential growth rate of the counting function (0.2.1)).

Corollary D (Corollary 9.2.11)**.** *Under the assumptions of Theorem C, there exists a strictly positive constant $\mathtt{C}$ such that for every t large enough one has*

$$\#\{\gamma\in\Gamma: \quad d_{\mathsf{X}_\mathrm{G}}(\mathsf{S}^o,\rho\gamma\cdot\mathsf{S}^o)\leq t\}\leq \mathtt{C}e^{\delta_\rho t}.$$

We finish this part of the introduction with a comment on the hypothesis over the limit set in Theorem C. Proposition 9.1.6 is not "intrinsic": the description of the limit cone $\mathscr{L}_\rho^{p,q}$ that we obtain strongly depends on the choice of a Weyl chamber of the system $\Sigma(\mathfrak{g},\mathfrak{a})$ contained in $\mathfrak{b}^+$. On the other hand, as in Proposition 3.2.4 one would like to relate the projection $b^o(g)$ with the Jordan projection of $\sigma^o(g^{-1})g$ and write

$$b^o(g)=\frac{1}{2}\lambda(\sigma^o(g^{-1})g), \tag{0.2.2}$$

and then try to estimate the right side of the last equality. However, this right side depends again on the choice of a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{b}^+$ (because the Jordan projection $\lambda(\cdot)$ does) and there seems to be no obvious choice to make. We will see that the assumption on the limit set in Theorem C canonically selects a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{b}^+$. Provided with this, the equation (0.2.2) can be assumed to hold and this is why we can obtain our result under this assumption (see the introduction of Part III for further comments on this point).

0.3 Final remarks

As a final comment, a word on the sets $\mathbf{\Omega}_\rho$ defined above for the pairs $(\mathrm{PSO}(p,q), \mathrm{PSO}(p,q-1))$ and $(\mathrm{PSL}(V), \mathrm{PSO}(p,q))$. Both constructions fit into a general framework which is the following. Let G be any connected semisimple Lie group with finite center and no compact factors. Fix a closed subgroup H of G and set

$$\mathsf{X} := \mathrm{G}/\mathrm{H}.$$

Fix a self opposite parabolic subgroup P of G (see Section 1.5 for a definition) and let $\mathsf{F} := \mathrm{G}/\mathrm{P}$ be the associated flag manifold of G. We emphazise that we do not assume that (G, H) is a symmetric pair, the sole assumption that we make here is that the quotient of $\mathsf{F} \times \mathsf{X}$ under the diagonal action of G is finite. For instance, if X is a flag manifold of G or a symmetric space of G this hypothesis is satisfied. For a point $o \in \mathsf{X}$, let H^o be its stabilizer in G and F^o be the union of open orbits of the action

$$\mathrm{H}^o \curvearrowright \mathsf{F}.$$

In joint work in progress with Florian Stecker we prove the following.

Theorem 0.3.1 (C.-Stecker)**.** *Let $\rho : \Gamma \to \mathrm{G}$ be a* P*-Anosov representation with limit map $\xi_\rho : \partial_\infty\Gamma \to \mathsf{F}$ and define*

$$\mathbf{\Omega}_\rho := \{o \in \mathsf{X} : \quad \xi_\rho(\partial_\infty\Gamma) \subset \mathsf{F}^o\}.$$

Then the action of Γ on $\mathbf{\Omega}_\rho$ induced by ρ is properly discontinuous.

For the symmetric pairs discussed above, the set $\mathbf{\Omega}_\rho$ coincides with the one of Theorem 0.3.1.

In order to prove Theorem 0.3.1 we develop a notion of *fat ideal* in the present setting that generalizes the notion introduced by Kapovich-Leeb-Porti [30]. Provided this notion, one can associate to a fat ideal a domain of discontinuity for ρ in the same way as Kapovich-Leeb-Porti do in [30]. Theorem 0.3.1 is a special case of this construction, as the set $\mathbf{\Omega}_\rho$ corresponds to a particular fat ideal in $\mathrm{G}\backslash(\mathsf{F} \times \mathsf{X})$. These constructions are described in a detailed way in Stecker's Ph.D Thesis [63] and outlined in Appendix C of this thesis.

Outline of the thesis

The thesis is structured in three parts:

- Part I treats preliminaries on symmetric spaces and Anosov representations. No original results are presented in this part, but we discuss in detail the Riemannian symmetric space of $\mathrm{PSO}(p,q)$ and of $\mathrm{PSL}(V)$, as well as the pseudo-Riemannian symmetric spaces $\mathsf{H}^{p,q-1}$ and $\mathsf{Q}_{p,q}$. Concrete ways of thinking the general structure theory of symmetric pairs will be proposed for these examples.

The original contributions of this thesis are detailed in Parts II and III. Each of these parts contains a little introduction with a reminder on what is proven there and also meant to exhibit the key results needed in each part.

- In Part II we prove our counting results for the symmetric pair

$$(\mathrm{PSO}(p,q), \mathrm{PSO}(p,q-1)).$$

- In Part III we prove our counting results for the symmetric pair

$$(\mathrm{PSL}(V), \mathrm{PSO}(p,q)).$$

We also include three appendices. In Appendix A we recall results on products of (r,ε)-*proximal* matrices. In Appendix B we briefly recall part of the theory of *metric Anosov flows*. In Appendix C we outline our joint results with Stecker.

Part I

Symmetric spaces and Anosov subgroups

Chapter 1

Symmetric spaces

We begin by recalling the definition and some basic facts about symmetric spaces associated to a (semisimple) Lie group. We will be mainly interested in three families of examples: Riemannian symmetric spaces, pseudo-Riemannian hyperbolic spaces and "spaces of quadratic forms". Even though we introduce the properties that we need using the general structure theory of semisimple Lie groups, we will use these three families as a source of examples and introduce some specific terminology for each of them that will be used throughout the thesis. We also include two sections (Sections 1.5 and 1.6) in which we recall the notion of *flag manifold* and the well known *Iwasawa decomposition* of G. Standard references for this chapter are the books of Helgason [27], Knapp [35], Kobayashi-Nomizu [36] and Schlichtkrull [62].

1.1 Definition and Lie theoretic generalities

Fix a connected semisimple Lie group G with finite center and no compact factors and let $\mathfrak{g}$ be its Lie algebra. The Killing form of $\mathfrak{g}$ will be denoted by

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}.$$

1.1.1 Definition

A G-*homogeneous space* is a smooth manifold X endowed with a transitive action of G by smooth diffeomorphisms. A *symmetric space* of G is a G-homogeneous space X for which point stabilizers are unions of connected components of

$$\mathrm{Fix}(\sigma) := \{g \in \mathrm{G} : \quad \sigma(g) = g\},$$

for some involutive automorphism σ of G.

Fix a point o in a symmetric space X and let H^o be its stabilizer in G. The associated involution of G is denoted by σ^o. The map

$$X \mapsto \frac{d}{dt}\Big|_{t=0} \exp(tX) \cdot o$$

gives a G-equivariant identification between

$$\mathfrak{q}^o := \{d\sigma^o = -1\}$$

and the tangent space to X at the point o. Let $\mathfrak{h}^o = \mathrm{Lie}(\mathrm{H}^o)$ be the Lie subalgebra of $\mathfrak{g}$ consisting on fixed points of $d\sigma^o$. One has the following decomposition

$$\mathfrak{g} = \mathfrak{h}^o \oplus \mathfrak{q}^o$$

of the Lie algebra $\mathfrak{g}$, which is orthogonal with respect to the Killing form κ. Thus, since κ is non degenerate, its restriction to $\mathfrak{q}^o$ is also non degenerate. The space X carries then a G-invariant non degenerate metric, called the *Killing metric*. Geodesics through the basepoint o for this metric are the curves of the form

$$t \mapsto \exp(tX) \cdot o$$

for some $X \in \mathfrak{q}^o$ (see Kobayashi-Nomizu [36, Theorem 3.2 of Ch. XI]). A submanifold of X is said to be *space-like* if the Killing metric is positive definite on its tangent spaces.

1.1.2 The Riemannian symmetric space

A *Cartan involution* of $\mathfrak{g}$ is an involutive automorphism

$$\tau : \mathfrak{g} \to \mathfrak{g}$$

for which the bilinear form on $\mathfrak{g}$ given by

$$(X, Y) \mapsto -\kappa(X, \tau \cdot Y)$$

is positive definite.

The stabilizer in G of a Cartan involution τ is compact. On the other hand, there exists an involutive automorphism σ^τ of G, whose derivative coincides with τ and for which the fixed point subgroup K^τ is a maximal compact subgroup of G (see Knapp [35, Theorem 6.31]). It follows that K^τ coincides with the stabilizer of τ in G.

Let X_G be the set of Cartan involutions of $\mathfrak{g}$. Since two elements of X_G always differ by the action of an element of G (c.f. [35, Corollary 6.19]), we obtain the identification

$$\mathsf{X}_\mathrm{G} \cong \mathrm{G}/\mathrm{K}^\tau$$

for any point τ in X_{G}. Therefore X_{G} is a symmetric space of G.

In this case we use the special notations

$$\mathfrak{p}^\tau := \{\tau = -1\} \text{ and } \mathfrak{k}^\tau := \{\tau = 1\},$$

and note that the Killing form is positive definite on $\mathfrak{p}^\tau$. Hence the Killing metric on X_{G} is Riemannian. The set X_{G} endowed with this metric will be called the *Riemannian symmetric space* of G. It is well known (see Helgason [27, Theorem 4.2 of Ch. IV]) that X_{G} is non positively curved.

Let $d_{\mathsf{X}_{\mathrm{G}}}(\cdot,\cdot)$ denote the (G-invariant) distance on X_{G} induced by the Riemannian structure. For every $X \in \mathfrak{p}^\tau$ one has

$$d_{\mathsf{X}_{\mathrm{G}}}(\tau, \exp(X)\cdot\tau) = \sqrt{\kappa(X,X)}. \tag{1.1.1}$$

1.1.3 Further Lie theoretic considerations

Fix any symmetric space X of G and a basepoint $o \in \mathsf{X}$. There exist Cartan involutions τ of $\mathfrak{g}$ for which σ^o and σ^τ commute and two of them always differ by the action of H^o_0, the connected component of H^o containing the identity element (see Matsuki [43, Lemmas 3 and 4]).

Fix a point $\tau \in \mathsf{X}_{\mathrm{G}}$ for which σ^τ and σ^o commute. Then one has the following decomposition

$$\mathfrak{g} = (\mathfrak{p}^\tau \cap \mathfrak{q}^o) \oplus (\mathfrak{p}^\tau \cap \mathfrak{h}^o) \oplus (\mathfrak{k}^\tau \cap \mathfrak{q}^o) \oplus (\mathfrak{k}^\tau \cap \mathfrak{h}^o)$$

of the Lie algebra $\mathfrak{g}$. In particular, we have

$$\mathfrak{q}^o = (\mathfrak{p}^\tau \cap \mathfrak{q}^o) \oplus (\mathfrak{k}^\tau \cap \mathfrak{q}^o)$$

and thus the signature of the Killing metric on X is $(\dim(\mathfrak{p}^\tau \cap \mathfrak{q}^o), \dim(\mathfrak{k}^\tau \cap \mathfrak{q}^o))$.

Let $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ be a (necessarily abelian) maximal subalgebra: two of them differ by the action of an element in $\mathrm{K}^\tau \cap \mathrm{H}^o$ (see Schlichtkrull [62, p.117]). Its dimension will be called the *rank* of X and for $X \in \mathfrak{b}$ we set

$$\|X\|_{\mathfrak{b}} := \sqrt{\kappa(X,X)}. \tag{1.1.2}$$

This defines a (positive definite) norm on $\mathfrak{b}$. The submanifold $\exp(\mathfrak{b})\cdot o$ of X is totally geodesic, space-like and flat (see [36, Theorem 4.3 and Proposition 4.4 of Ch. XI]).

Remark 1.1.1. In the special case $\mathsf{X} = \mathsf{X}_{\mathrm{G}}$, we will use the special notation $\mathfrak{a}$ instead of $\mathfrak{b}$ for a maximal subalgebra of $\mathfrak{p}^\tau$. These subalgebras are called *Cartan subspaces* of $\mathfrak{g}$.

◇

1.2 Examples

We now discuss two families of examples of symmetric spaces. The first family is associated with the projectivized special orthogonal group $\mathrm{PSO}(p,q)$, for positive integers p and q. The second one is associated with the projectivized special linear group $\mathrm{PSL}(V)$, for a real vector space of dimension ≥ 2. In each case we provide concrete descriptions of the Riemannian symmetric space and introduce certain families of pseudo-Riemannian symmetric spaces. We also compute their rank and provide explicit descriptions of the Lie subalgebras $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$.

1.2.1 Two symmetric spaces associated to $\mathrm{G} = \mathrm{PSO}(p,q)$

Fix two integers $p,q \geq 1$ and let $d := p+q$. We assume here that $d > 2$. Denote by $\mathbb{R}^{p,q}$ the vector space $\mathbb{R}^d$ endowed with the symmetric bilinear form

$$\langle (x_1,\dots,x_d),(y_1,\dots,y_d)\rangle_{p,q} := \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{d} x_i y_i$$

of signature (p,q). Let $\mathrm{PSO}(p,q)$ be the subgroup of $\mathrm{PSL}_d(\mathbb{R})$ consisting on elements whose lifts to $\mathrm{SL}_d(\mathbb{R})$ preserve this form. Throughout the thesis, the notation $\mathrm{PSO}(p,q)$ will always mean that all these choices have been made.

Until otherwise stated, denote $\mathrm{G} := \mathrm{PSO}(p,q)$. The Killing form of $\mathfrak{g}$ is given in this case by

$$\kappa(X,Y) = (d-2)\mathrm{tr}(XY)$$

(see Helgason [27, p.180 & p.189]).

For a subspace π of $\mathbb{R}^{p,q}$ we denote by $\pi^{\perp_{p,q}}$ its orthogonal complement with respect to $\langle\cdot,\cdot\rangle_{p,q}$, i.e.

$$\pi^{\perp_{p,q}} := \{\hat{x} \in \mathbb{R}^{p,q} : \quad \langle \hat{x},\hat{y}\rangle_{p,q} = 0 \text{ for all } \hat{y} \in \pi\}.$$

The Riemannian symmetric space of G

The space X_{G} can be identified in this case with the space of q-dimensional subspaces of $\mathbb{R}^{p,q}$ on which the form $\langle\cdot,\cdot\rangle_{p,q}$ is negative definite. Explicitly, the identification is as follows. Given a q-dimensional negative definite subspace τ, one defines the inner product

$$\langle\cdot,\cdot\rangle_\tau \qquad (1.2.1)$$

of $\mathbb{R}^d$ that coincides with $-\langle\cdot,\cdot\rangle_{p,q}$ (resp. $\langle\cdot,\cdot\rangle_{p,q}$) on τ (resp. $\tau^{\perp_{p,q}}$) and for which τ and $\tau^{\perp_{p,q}}$ are orthogonal. It can be seen that the adjoint operator $^{\star_\tau}\cdot$ associated to this inner product preserves the group G. The involution $d\sigma^\tau$, for

$$\sigma^\tau : \mathrm{G} \to \mathrm{G} : \quad \sigma^\tau(g) := {}^{\star_\tau}g^{-1},$$

is a Cartan involution of $\mathfrak{g}$. The group K^τ is in this case isomorphic to $\mathrm{PS}(\mathrm{O}(p) \times \mathrm{O}(q))$.

Example 1.2.1. Cartan subspaces can be described as follows. Let $\tau \in \mathsf{X}_\mathrm{G}$, which we think as a negative definite q-dimensional subspace of $\mathbb{R}^{p,q}$. Since $\tau^{\perp_{p,q}}$ is linearly disjoint from τ we can define the linear transformation[1]

$$J^\tau := \mathrm{id}_\tau \oplus (-\mathrm{id}_{\tau^{\perp_{p,q}}}).$$

Let l be the minimum between p and q and pick a maximal collection $\mathcal{C}$ of pairwise $\langle\cdot,\cdot\rangle_{p,q}$-orthogonal J^τ-invariant 2-dimensional subspaces of $\mathbb{R}^{p,q}$ on which the restriction of the form $\langle\cdot,\cdot\rangle_{p,q}$ has signature $(1,1)$ (note that $\mathcal{C}$ contains l elements). Define $\mathfrak{a}_\mathcal{C}$ to be the subset of $\mathfrak{g}$ consisting on elements that preserve each element of $\mathcal{C}$ and such that in the orthogonal complement of span $\mathcal{C}$ are equal to zero. Then $\mathfrak{a}_\mathcal{C}$ is contained in $\mathfrak{p}^\tau$ and it is a Cartan subspace of $\mathfrak{g}$. Conversely, every Cartan subspace of $\mathfrak{g}$ arises in this way.

◇

The pseudo-Riemannian hyperbolic space $\mathsf{H}^{p,q-1}$

For a vector $\hat{x}$ in $\mathbb{R}^{p,q}$ denote by $x = [\hat{x}]$ its class in the projective space $\mathsf{P}(\mathbb{R}^{p,q})$ of $\mathbb{R}^{p,q}$. Set

$$\mathsf{H}^{p,q-1} := \{o = [\hat{o}] \in \mathsf{P}(\mathbb{R}^{p,q}) : \quad \langle \hat{o}, \hat{o}\rangle_{p,q} < 0\}.$$

Note that G acts transitively on $\mathsf{H}^{p,q-1}$. Fix a point o in $\mathsf{H}^{p,q-1}$ and consider the matrix

$$J^o := \mathrm{id}_o \oplus \left(-\mathrm{id}_{o^{\perp_{p,q}}}\right).$$

One can see that the stabilizer in G of o equals the fixed point subgroup of the involution σ^o of G given by

$$\sigma^o(g) := J^o g J^o \tag{1.2.2}$$

and therefore $\mathsf{H}^{p,q-1}$ is a symmetric space of G. This space is called the *pseudo-Riemannian hyperbolic space of signature* $(p, q-1)$. Note that when $q = 1$ one has the equality

$$\mathsf{H}^{p,q-1} = \mathsf{X}_\mathrm{G}.$$

[1] Note that the subgroup K^τ can be now described as the set of fixed points of the involution $g \mapsto J^\tau g J^\tau$ of G.

For this reason, from now on the notation $\mathsf{H}^{p,q-1}$ will always mean that q is strictly larger than 1.

Remark 1.2.2. There exists a G-equivariant identification

$$T_o\mathsf{H}^{p,q-1} \cong o^{\perp_{p,q}}$$

for tangent spaces to $\mathsf{H}^{p,q-1}$. Hence, one can endow $\mathsf{H}^{p,q-1}$ with a G-invariant metric coming from restriction of the form $\langle\cdot,\cdot\rangle_{p,q}$ to $o^{\perp_{p,q}}$ for all $o \in \mathsf{H}^{p,q-1}$. It can be seen that this metric is a positive multiple of the Killing metric (this is an explicit computation that we omit). The precise scaling factor can be found in C. [14, Remark 2.3]. We will sometimes abuse of notations and consider these two metrics as being the same whenever convenient.

◇

Remark 1.2.3. Let $o \in \mathsf{H}^{p,q-1}$. Then the action of H_0^o (the connected component of H^o containing the identity element) on $T_o\mathsf{H}^{p,q-1}$ is conjugate to the action of $\mathrm{SO}(p, q-1)$ on $\mathbb{R}^{p,q-1}$.

◇

We now give a concrete description of a subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ in this special case.

Example 1.2.4. Let o be a point in $\mathsf{H}^{p,q-1}$ and τ be a point of X_{G}. Suppose further that the line o is contained in the subspace τ. Then the involution σ^τ commutes with σ^o.

Pick a line ℓ in $\tau^{\perp_{p,q}}$ and define $\mathfrak{b}$ to be the set of elements X in $\mathfrak{g}$ whose restriction to the subspace $\ell \oplus o$ is diagonalizable and such that

$$(\ell \oplus o)^{\perp_{p,q}} \subset \ker(X).$$

It can be seen that $\mathfrak{b}$ is a maximal subalgebra in $\mathfrak{p}^\tau \cap \mathfrak{q}^o$. The subspace $\exp(\mathfrak{b}) \cdot o$ is in this case a (space-like) geodesic.

◇

The following remark will be used repeatedly Part II.

Remark 1.2.5. Even though G does not act on $\mathbb{R}^d$, it makes sense to ask if an element g of G preserves a norm on $\mathbb{R}^d$ (this notion does not depend on the choice of a lift of g to $\mathrm{SL}_d(\mathbb{R})$). Let $o \in \mathsf{H}^{p,q-1}$ and $\tau \in \mathsf{X}_{\mathrm{G}}$ such that $o \subset \tau$. Let $\langle\cdot,\cdot\rangle_\tau$ be the inner product of $\mathbb{R}^d$ defined as in (1.2.1) and $\|\cdot\|_\tau$ be the associated norm, which is K^τ-invariant. One can see that this norm is preserved by the matrix J^o.

◇

1.2.2 Spaces of quadratic forms of fixed signature

Let $d \geq 2$ be an integer and V be a real vector space of dimension d. Throughout the thesis, the notation $\mathrm{PSL}(V)$ will always mean that all these choices have been made.

In this subsection we denote $\mathrm{G} := \mathrm{PSL}(V)$. The vector space consisting on quadratic forms on V is denoted by $\mathcal{Q}(V)$. Note that G acts on the space of rays of $\mathcal{Q}(V)$ and each open orbit of this action consists on the set of non degenerate quadratic forms of a given signature. For non negative integers p and q satisfying $d = p+q$, the orbit consisting on quadratic forms of signature[2] (p, q) is denoted by $\mathsf{Q}_{p,q}$.

The goal now is to show that $\mathsf{Q}_{p,q}$ is a symmetric space of G. In order to do this, we fix a point o in $\mathsf{Q}_{p,q}$ and let $\langle \cdot, \cdot \rangle_o$ be the bilinear form associated to a representative of o. We will say in short that the form $\langle \cdot, \cdot \rangle_o$ is a *representative* of o. Since o is non degenerate, we can define the *o-adjoint operator*

$$^{\star_o}\cdot : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$$

where, for $T \in \mathfrak{gl}(V)$, $^{\star_o}T$ is the unique linear transformation of V that satisfies the equality

$$\langle T \cdot u, v \rangle_o = \langle u, {}^{\star_o}T \cdot v \rangle_o$$

for all u and v in V. Note that the o-adjoint $^{\star_o}T$ does not depend on the choice of the representative $\langle \cdot, \cdot \rangle_o$ of o. Moreover, $^{\star_o}\cdot$ preserves $\mathrm{SL}(V)$ and descends to a map $\mathrm{G} \to \mathrm{G}$, that we still denote by $^{\star_o}\cdot$. Define σ^o to be the involutive automorphism of G given by

$$\sigma^o(g) := {}^{\star_o}g^{-1}.$$

Then $\mathsf{Q}_{p,q}$ identifies with G/H^o, where $\mathrm{H}^o \cong \mathrm{PSO}(p, q)$ is the subgroup of G consisting on fixed points of the involution σ^o. We conclude that $\mathsf{Q}_{p,q}$ is a symmetric space of G.

We now explicitly describe a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$. In order to do that, we fix some terminology that will remain valid for the rest of the thesis. For a subspace π of V denote by $\pi^{\perp_o}$ its orthogonal complement with respect to the form $\langle \cdot, \cdot \rangle_o$. A set is said to be *o-orthogonal* if its elements are pairwise $\langle \cdot, \cdot \rangle_o$-orthogonal. The *o-sign* of a vector v in V is defined by

$$\mathrm{sg}_o(v) := \begin{cases} 1 & \text{if } \langle v, v \rangle_o > 0 \\ -1 & \text{if } \langle v, v \rangle_o < 0 \\ 0 & \text{if } \langle v, v \rangle_o = 0 \end{cases}.$$

[2] That is, forms that admit a diagonal expression with p (resp. q) positive (resp. negative) eigenvalues.

A vector v in V is said to be $\langle\cdot,\cdot\rangle_o$-*unitary* if

$$\langle v, v\rangle_o \in \{-1, 1\}.$$

A basis $\mathcal{B}$ of V is said to be $\langle\cdot,\cdot\rangle_o$-*orthonormal* if it is o-orthogonal and its elements are $\langle\cdot,\cdot\rangle_o$-unitary. Finally, the *o-sign* of a line ℓ in V is defined in an analogous way and a *basis of lines* of V is a set of d lines in V that span V.

Example 1.2.6. Let $\mathcal{B}$ be a $\langle\cdot,\cdot\rangle_o$-orthonormal basis of V and $\langle\cdot,\cdot\rangle$ be the inner product of V for which this basis is orthonormal. The associated point in X_{G} will be denoted by τ and we emphasize the link between the inner product $\langle\cdot,\cdot\rangle$ and the point τ by denoting $\langle\cdot,\cdot\rangle_\tau := \langle\cdot,\cdot\rangle$. It can be seen that σ^o and σ^τ commute.

Pick $\mathfrak{b}$ to be the subset of $\mathfrak{g}$ consisting on elements which are diagonal in the basis $\mathcal{B}$. Since $\mathcal{B}$ is both $\langle\cdot,\cdot\rangle_o$-orthonormal and $\langle\cdot,\cdot\rangle_\tau$-orthonormal, one has

$$\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$$

and one can see that $\mathfrak{b}$ is a maximal subalgebra[3] in $\mathfrak{p}^\tau \cap \mathfrak{q}^o$. Note that in this case $\mathfrak{b}$ is also a maximal subalgebra in $\mathfrak{p}^\tau$ and therefore we can denote $\mathfrak{b} = \mathfrak{a}$ whenever convenient.

On the other hand, one has the equality

$$\exp(\mathfrak{b}) \cdot o = \{o' \in \mathsf{Q}_{p,q} : \ \mathcal{B} \text{ is } o'\text{-orthogonal and } \mathrm{sg}_o(v) = \mathrm{sg}_{o'}(v) \ \ \forall v \in \mathcal{B}\}.$$

The submanifold $\exp(\mathfrak{b}) \cdot o$ is then a totally geodesic isometric embedding of $\mathbb{R}^{d-1}$ inside $\mathsf{Q}_{p,q}$ or, in short, a *space-like flat.*

◇

Remark 1.2.7. Note that one has natural identifications

$$\mathsf{Q}_{0,d} \cong \mathsf{Q}_{d,0} \cong \mathsf{X}_{\mathrm{G}}.$$

Indeed, to an element $\tau \in \mathsf{Q}_{d,0}$ one asigns the Cartan involution $d\sigma^\tau$ of $\mathfrak{g}$. For this reason, from now on the notation $\mathsf{Q}_{p,q}$ will always mean that p and q are strictly positive.

Cartan subspaces of $\mathfrak{g}$ contained in $\mathfrak{p}^\tau$ (for a given basepoint $\tau \in \mathsf{X}_{\mathrm{G}}$) bijectively correspond to τ-orthogonal bases of lines of V.

◇

[3] Conversely, a maximal subalgebra contained in $\mathfrak{p}^\tau \cap \mathfrak{q}^o$ determines a basis of lines $\mathcal{C}$ of V which is both o-orthogonal and τ-orthogonal.

1.3 The submanifold S^o

We now come back to the case in which G is any semisimple Lie group with finite center and no compact factors and we fix a symmetric space X of G. To each point $o \in \mathsf{X}$ one associates a totally geodesic copy S^o of the Riemannian symmetric space of H^o inside X_G as follows.

Given a point o in X let

$$\mathsf{S}^o := \{\tau \in \mathsf{X}_\mathrm{G} : \quad \sigma^\tau \sigma^o = \sigma^o \sigma^\tau\}.$$

The submanifold S^o will play a central role in our thesis, because it will allow us to define certain geometric quantities in X_G from which we build our counting functions. These quantities will have a geometric counterpart in the space X.

Note that S^o is H^o-invariant and because of Matsuki [43, Lemmas 3 and 4] one has

$$\mathsf{S}^o = \mathrm{H}^o_0 \cdot \tau$$

for any $\tau \in \mathsf{S}^o$. In particular, the tangent space to S^o at τ identifies with

$$\mathfrak{p}^\tau \cap \mathfrak{h}^o$$

and S^o is totally geodesic (see Helgason [27, Theorem 7.2 of Ch. IV]). Further, it is a copy of the Riemannian symmetric space[4] of H^o_0.

Example 1.3.1.

- When $\mathsf{X} = \mathsf{X}_\mathrm{G}$, the set S^o is reduced to the single point o.
- Let o be a point in $\mathsf{H}^{p,q-1}$. A negative definite q-dimensional subspace τ belongs to S^o if and only if it contains the line o. We see then that the isometry $\mathsf{S}^o \to \mathsf{X}_{\mathrm{PSO}(p,q-1)}$ is given by

$$\tau \mapsto \tau \cap o^{\perp_{p,q}}.$$

- Let o be a point in $\mathsf{Q}_{p,q}$. An homothety class of inner products τ belongs to S^o if and only if there exist representatives $\langle \cdot, \cdot \rangle_o$ of o and $\langle \cdot, \cdot \rangle_\tau$ of τ and a basis $\mathcal{B}$ of V which is both $\langle \cdot, \cdot \rangle_o$-orthonormal and $\langle \cdot, \cdot \rangle_\tau$-orthonormal. We see then that the isometry $\mathsf{S}^o \to \mathsf{X}_{\mathrm{PSO}(p,q)}$ is given by the map that sends each $\tau \in \mathsf{S}^o$ to the subspace of V spanned by the vectors of $\mathcal{B}$ which are negative for the form o.

[4] Note that, rather than semisimple, the Lie subgroup H^o_0 is *reductive*: since the Killing form κ restricts to a non degenerate H^o_0-invariant symmetric bilinear form on $\mathfrak{h}^o$, the Lie algebra $\mathfrak{h}^o$ splits as a direct sum of its center and a semisimple Lie algebra. As in the semisimple case, one can develop a theory of Cartan involutions and Cartan decomposition in the setting of reductive Lie groups (see Knapp [35, Chapter 7]). The submanifold S^o identifies then with the space of Cartan involutions of H^o_0. We have decided in this thesis to focus on semisimple Lie groups: in the two examples we deal with, namely $\mathsf{X} = \mathsf{H}^{p,q-1}$ and $\mathsf{X} = \mathsf{Q}_{p,q}$, the subgroup H^o_0 is semisimple.

◇

A geometric interpretation in X_G of the maximal subalgebras $\mathfrak{b}$ of $\mathfrak{p}^\tau \cap \mathfrak{q}^o$ is provided by the following lemma.

Lemma 1.3.2. *Fix a point τ in S^o and a maximal subalgebra $\mathfrak{b}$ contained in $\mathfrak{p}^\tau \cap \mathfrak{q}^o$. Then*

$$\exp(\mathfrak{b}) \cdot \tau$$

is a totally geodesic flat subspace of X_G, orthogonal to S^o at τ and isometric to $\mathbb{R}^{\dim \mathfrak{b}}$. It is maximal dimensional for these properties.

Here by "orthogonal to S^o at τ" we mean that the tangent space of $\exp(\mathfrak{b}) \cdot \tau$ at τ is contained in the orthogonal complement of $T_\tau \mathsf{S}^o$.

Proof of Lemma 1.3.2. It is well known that $\exp(\mathfrak{b}) \cdot \tau$ is totally geodesic and isometric to $\mathbb{R}^{\dim \mathfrak{b}}$. Since $T_\tau \mathsf{S}^o = \mathfrak{p}^\tau \cap \mathfrak{h}^o$, it is orthogonal to S^o at τ.

It remains to show maximality, but this follows from the fact that maximal totally geodesic flat subspaces of X_G through τ bijectively correspond to maximal subalgebras of $\mathfrak{p}^\tau$. □

1.4 Polar, Cartan and Jordan projections

In this section we recall the well known *polar decomposition* of G associated to a symmetric space X, which generalizes the classical *Cartan decomposition* of G and that is well adapted to the study of the geometry of X. We discuss these geometric interpretations and establish some basic relations between the two decompositions, which are friutful to study the polar projection by means of the Cartan projection.

Fix an element τ in S^o and a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$.

1.4.1 Weyl chambers

Let

$$\mathfrak{g}^{\tau o} := (\mathfrak{p}^\tau \cap \mathfrak{q}^o) \oplus (\mathfrak{k}^\tau \cap \mathfrak{h}^o)$$

be the Lie algebra consisting on fixed points of the involution $d\sigma^\tau d\sigma^o$. A non zero functional $\alpha \in \mathfrak{b}^*$ is called a *restricted root of $\mathfrak{b}$ in $\mathfrak{g}^{\tau o}$* if the subspace

$$\mathfrak{g}^{\tau o}_\alpha := \{Y \in \mathfrak{g}^{\tau o} : \quad [X, Y] = \alpha(X)Y \text{ for all } X \in \mathfrak{b}\}$$

is non zero. In that case, $\mathfrak{g}^{\tau o}_\alpha$ is called the associated *root space*. The set of restricted roots of $\mathfrak{b}$ in $\mathfrak{g}^{\tau o}$ is denoted by $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$ (c.f. Schlichtkrull [62, p. 117]).

Let $\mathfrak{b}^+$ be the closure of a connected component of

$$\mathfrak{b} \setminus \bigcup_{\alpha \in \Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})} \ker(\alpha).$$

This is called a *closed Weyl chamber* of the system $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$. Let $\Sigma^+(\mathfrak{g}^{\tau o}, \mathfrak{b})$ be the corresponding positive system, that is, the set of restricted roots in $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$ which are non negative on $\mathfrak{b}^+$.

Before proceeding with the general theory, let us discuss some examples.

Weyl chambers in Riemannian symmetric spaces

When $\mathsf{X} = \mathsf{X}_{\mathrm{G}}$ the Lie algebra $\mathfrak{g}^{\tau o}$ coincides with $\mathfrak{g}$ (because o and τ must coincide) and $\mathfrak{b} = \mathfrak{a}$ is a Cartan subspace of $\mathfrak{g}$. In this case we denote by Σ the set of restricted roots $\Sigma(\mathfrak{g}, \mathfrak{a})$ and by $\mathfrak{a}^+$ a closed Weyl chamber. The corresponding positive system is denoted by Σ^+.

Example 1.4.1. Suppose that $\mathrm{G} = \mathrm{PSO}(p, q)$ and let $\mathfrak{a}$ be given by a maximal collection $\mathcal{C}$ of $\langle \cdot, \cdot \rangle_{p,q}$-orthogonal J^τ-invariant subspaces of $\mathbb{R}^{p,q}$ of signature $(1, 1)$ (c.f. Subsection 1.2.1). Pick an isotropic line ℓ_ω in each element $\omega \in \mathcal{C}$ and let $\varepsilon_\omega \in \mathfrak{a}^*$ be the map that assigns to each element $X \in \mathfrak{a}$ its eigenvalue in the line ℓ_ω. In this case one has

$$\Sigma = \{\pm\varepsilon_\omega \pm \varepsilon_{\omega'} : \quad \omega \neq \omega' \text{ in } \mathcal{C}\}$$

if $p = q$ and

$$\Sigma = \{\pm\varepsilon_\omega \pm \varepsilon_{\omega'} : \quad \omega \neq \omega' \text{ in } \mathcal{C}\} \cup \{\pm\varepsilon_\omega : \quad \omega \in \mathcal{C}\}$$

if $p \neq q$. The choice of a closed Weyl chamber corresponds then to the choice of a total order

$$\mathcal{C} = \{\omega_1, \dots, \omega_l\}$$

on $\mathcal{C}$ and the choice of an isotropic line ℓ_j in each $\omega_j \in \mathcal{C}$. If these choices are given, we set

$$\varepsilon_j := \varepsilon_{\omega_j}$$

for each $j = 1, \dots, l = \min\{p, q\}$.

◇

Example 1.4.2. Suppose that $\mathrm{G} = \mathrm{PSL}(V)$ and that $\mathfrak{a}$ is given by τ-orthogonal basis of lines $\mathcal{C}$ of V (Remark 1.2.7). Given a line $\ell \in \mathcal{C}$, let $\varepsilon_\ell \in \mathfrak{a}^*$ be the map that assigns to each element $X \in \mathfrak{a}$ its eigenvalue in the line ℓ. For different ℓ and ℓ' in $\mathcal{C}$, set

$$\alpha_{\ell\ell'}(X) := \varepsilon_\ell(X) - \varepsilon_{\ell'}(X).$$

The set of restricted roots is in this case

$$\Sigma = \{\alpha_{\ell\ell'} : \quad \ell \neq \ell' \text{ in } \mathcal{C}\}.$$

The choice of a closed Weyl chamber corresponds then to the choice of a total order

$$\mathcal{C} = \{\ell_1, \dots, \ell_d\}$$

on $\mathcal{C}$. If this choice is given, we set

$$\varepsilon_j := \varepsilon_{\ell_j} \text{ and } \alpha_{ji} := \varepsilon_j - \varepsilon_i$$

for each j different from i in $\{1, \dots, d\}$.

◇

Weyl chambers for pseudo-Riemannian hyperbolic spaces

Consider the case $\mathsf{X} = \mathsf{H}^{p,q-1}$ and $\mathfrak{b}$ constructed from the choice of a line ℓ in $\tau^{\perp_{p,q}}$ (see Example 1.2.4). A closed Weyl chamber $\mathfrak{b}^+$ is given in this case by the choice of a ray in $\mathfrak{b}$ starting at the origin. This is equivalent to the choice of an isotropic line in the subspace $\ell \oplus o$.

Weyl chambers for spaces of quadratic forms

Consider now the case $\mathsf{X} = \mathsf{Q}_{p,q}$ and fix an o-orthogonal and τ-orthogonal basis of lines $\mathcal{C}$ of V. Let $\mathfrak{b} = \mathfrak{a}$ be the associated Cartan subspace of $\mathfrak{g}$, which is a maximal subalgebra both in $\mathfrak{p}^\tau \cap \mathfrak{q}^o$ and $\mathfrak{p}^\tau$ (c.f. Example 1.2.6). It can be seen that one has the equality

$$\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b}) = \{\alpha_{\ell\ell'} \in \Sigma(\mathfrak{g}, \mathfrak{a}) : \quad \ell \neq \ell' \text{ in } \mathcal{C} \text{ and } \operatorname{sg}_o(\ell) = \operatorname{sg}_o(\ell')\}.$$

Indeed, recall from Example 1.4.2 that a non zero functional $\alpha \in \mathfrak{b}^*$ belongs to $\Sigma(\mathfrak{g}, \mathfrak{a})$ if and only if there exist two lines $\ell \neq \ell'$ in $\mathcal{C}$ such that

$$\alpha_{\ell\ell'} = \alpha.$$

The associated root space

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} : \quad [X, Y] = \alpha(X)Y \text{ for all } X \in \mathfrak{b}\}$$

intersects the Lie subalgebra $\mathfrak{g}^{\tau o}$ if and only if $\operatorname{sg}_o(\ell)$ coincides with $\operatorname{sg}_o(\ell')$.

An explicit way of prescribing a Weyl chamber $\mathfrak{b}^+$ is the following. Write $\mathcal{C} = \mathcal{C}^+ \sqcup \mathcal{C}^-$ where $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) is the subset of $\mathcal{C}$ consisting on o-positive definite (resp. o-negative definite) lines. Fix total orders

$$\mathcal{C}^+ = \{\ell_1^+, \ldots, \ell_p^+\} \text{ and } \mathcal{C}^- = \{\ell_1^-, \ldots, \ell_q^-\}$$

on $\mathcal{C}^+$ and $\mathcal{C}^-$. Then a positive system in $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$ is given by

$$\Sigma^+(\mathfrak{g}^{\tau o}, \mathfrak{b}) := \left\{\alpha_{\ell_j^+ \ell_i^+} : \quad 1 \leq j < i \leq p\right\} \sqcup \left\{\alpha_{\ell_j^- \ell_i^-} : \quad 1 \leq j < i \leq q\right\}$$

and the corresponding closed Weyl chamber is

$$\mathfrak{b}^+ := \{X \in \mathfrak{b} : \quad \alpha(X) \geq 0 \text{ for all } \alpha \in \Sigma^+(\mathfrak{g}^{\tau o}, \mathfrak{b})\}.$$

Conversely, the choice of a Weyl chamber $\mathfrak{b}^+$ for the system $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$ induces total orders on the sets $\mathcal{C}^{\pm}$.

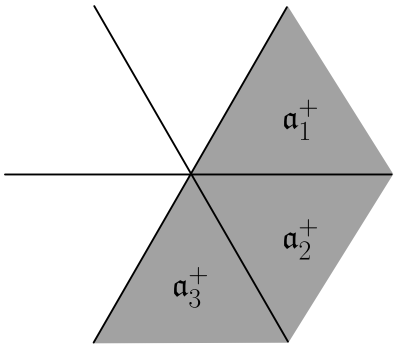


Figure 1.1: Weyl chambers for $\mathsf{Q}_{2,1}$. In light grey, a Weyl chamber $\mathfrak{b}^+$ of the system $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$. This Weyl chamber is a union of three Weyl chambers $\mathfrak{a}_1^+$, $\mathfrak{a}_2^+$ and $\mathfrak{a}_3^+$ of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$.

1.4.2 Polar projection

We now go back to the general theory. Fix a closed Weyl chamber $\mathfrak{b}^+$ for the system $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$. It is well known (see e.g. Schlichtkrull [62, Proposition 7.1.3]) that the *polar decomposition* of G holds:

$$\mathrm{G} = \mathrm{K}^\tau \exp(\mathfrak{b}^+)\mathrm{H}^o,$$

where the $\exp(\mathfrak{b}^+)$-component is uniquely determined and one can define

$$b^{o,\tau} : \mathrm{G} \to \mathfrak{b}^+$$

by taking the log of this component. This is a continuous map called the *polar projection* of G associated to the choice of o, τ and $\mathfrak{b}^+$.

Remark 1.4.3. Note that $b^{o,\tau}$ descends to a map $\mathsf{X} \cong \mathrm{G}/\mathrm{H}^o \to \mathfrak{b}^+$ which, by definition, is proper.

◇

Cartan and Jordan projections

Observe that in the case $\mathsf{X} = \mathsf{X}_{\mathrm{G}}$, the map $b^{o,\tau}$ coincides with the well known *Cartan projection*, associated to the *Cartan decomposition*

$$\mathrm{G} = \mathrm{K}^\tau \exp(\mathfrak{a}^+)\mathrm{K}^\tau$$

of G. We will use the notation

$$a^\tau : \mathrm{G} \to \mathfrak{a}^+$$

for this map in this special case. Recall that one has the *Jordan projection*

$$\lambda : \mathrm{G} \to \mathfrak{a}^+$$

of G, which is characterized by the formula

$$\lambda(g) = \lim_{n\to\infty} \frac{a^\tau(g^n)}{n} \tag{1.4.1}$$

for every g in G.

Remark 1.4.4. Suppose that $\mathrm{G} = \mathrm{PSO}(p, q)$ or $\mathrm{G} = \mathrm{PSL}(V)$. For $j = 1, \ldots, \min\{p, q\}$ in the first case or for $j = 1, \ldots, d$ in the second one, we will use the notations

$$a_j^\tau := \varepsilon_j \circ a^\tau \text{ and } \lambda_j := \varepsilon_j \circ \lambda$$

(c.f. Examples 1.4.1 and 1.4.2). Note that in both cases one has that $a_1^\tau(g)$ equals the logarithm of the operator norm of g (computed with respect to the inner product τ) and that $\lambda_1(g)$ is the logarithm of the spectral radius of g.

◇

Relation between polar and Cartan projections

When X is different from the Riemannian symmetric space of G we still have a relation between polar and Cartan projections, which is the following. Let $\mathfrak{a} \subset \mathfrak{p}^\tau$ be a maximal subalgebra containing $\mathfrak{b}$. Let $\mathrm{W} = \mathrm{W}(\mathfrak{g}, \mathfrak{a})$ be the *Weyl group* of the pair $(\mathfrak{g}, \mathfrak{a})$. It is defined by the equality

$$\mathrm{W} := \hat{\mathrm{W}}/\mathrm{M}$$

where $\hat{\mathrm{W}} = \mathrm{N}_{\mathrm{K}^\tau}(\mathfrak{a})$ (resp. $\mathrm{M} = \mathrm{Z}_{\mathrm{K}^\tau}(\mathfrak{a})$) is the normalizer (resp. centralizer) of $\mathfrak{a}$ in K^τ, and naturally acts on $\mathfrak{a}$ by linear isometries of the norm $\| \cdot \|_{\mathfrak{a}}$. This action induces an action on the set of Weyl chambers of the system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ which is simply transitive.

Fix a Weyl chamber $\mathfrak{a}^+$ for the system Σ and let g be any element of G. Write

$$g = k \exp(b^{o,\tau}(g)) h$$

in polar coordinates. Since $d\sigma^o$ acts as $-\mathrm{id}$ on $\mathfrak{b}$ we have

$$g\sigma^o(g^{-1}) = k \exp(2b^{o,\tau}(g))\sigma^o(k^{-1}).$$

Observe that σ^o preserves K^τ and therefore we find an element $w_g \in \mathrm{W}$ for which one has

$$b^{o,\tau}(g) = \tfrac{1}{2} w_g \cdot a^\tau(g\sigma^o(g^{-1})).$$

In particular, the following equality holds for any $g \in \mathrm{G}$:

$$\|b^{o,\tau}(g)\|_{\mathfrak{b}} = \|b^{o,\tau}(g)\|_{\mathfrak{a}} = \tfrac{1}{2}\|a^\tau(g\sigma^o(g^{-1}))\|_{\mathfrak{a}}.$$

More concretely we have the following.

Example 1.4.5.

- Let $\mathsf{X} = \mathsf{H}^{p,q-1}$ and $\| \cdot \|_\tau$ be the norm on $\mathbb{R}^d$ induced by the choice of τ. Then for every g in $\mathrm{PSO}(p,q)$ one has[5]

  $$\|b^{o,\tau}(g)\|_{\mathfrak{b}} = \tfrac{1}{2} \log \|gJ^o g^{-1} J^o\|_\tau.$$

  Note also that thanks to Remark 1.2.5 one also has

  $$\|b^{o,\tau}(g)\|_{\mathfrak{b}} = \tfrac{1}{2} \log \|J^o g J^o g^{-1}\|_\tau.$$

- Let $\mathsf{X} = \mathsf{Q}_{p,q}$. Then for every $g \in \mathrm{PSL}(V)$, the number $\|b^{o,\tau}(g)\|_{\mathfrak{b}}$ can be interpreted in terms of the *singular values* (computed with respect to τ) of the element $g\sigma^o(g^{-1})$ (see Horn-Johnson [28, Section 7.3 of Chapter 7] for the definition of singular values of an element in $\mathrm{PSL}(V)$).

◇

[5]Here we are abusing notations, because the equality actually holds up to a scalar positive constant only depending on $d = p + q$ (c.f. Remark 1.2.2).

Geometric interpretation of polar projection

We now discuss geometric interpretations of the polar projection $b^{o,\tau}$. We will assume here that X is not the Riemannian symmetric space. Indeed when $\mathsf{X} = \mathsf{X}_{\mathrm{G}}$, because of equations (1.1.1) and (1.1.2) one has

$$d_{\mathsf{X}_{\mathrm{G}}}(\tau, g \cdot \tau) = \|a^\tau(g)\|_{\mathfrak{a}}$$

for every $g \in \mathrm{G}$ and the geometric interpretation is straightforward in this case. The following generalization of the previous equality is inspired by the work of Oh-Shah [46], which is concerned with the symmetric pair $(\mathrm{PSO}(1,3), \mathrm{PSO}(1,2))$ (c.f. Figure 1.2 below).

Proposition 1.4.6. *For every g in G one has*

$$\|b^{o,\tau}(g)\|_{\mathfrak{b}} = d_{\mathsf{X}_{\mathrm{G}}}(\tau, g \cdot \mathsf{S}^o).$$

Proof. The function $g \mapsto d_{\mathsf{X}_{\mathrm{G}}}(g^{-1} \cdot \tau, \mathsf{S}^o)$ is K^τ-invariant on the left and H^o-invariant on the right, hence it suffices to check that the equality of the statement holds when $g = \exp(X)$ for some $X \in \mathfrak{b}^+$.

Since X_{G} is non positively curved, there exists a unique geodesic through $g^{-1} \cdot \tau = \exp(-X) \cdot \tau$ which is orthogonal to S^o. By Lemma 1.3.2 this geodesic is $\exp(\mathbb{R}X) \cdot \tau$ and intersects S^o at τ, hence

$$d_{\mathsf{X}_{\mathrm{G}}}(\exp(-X) \cdot \tau, \mathsf{S}^o) = d_{\mathsf{X}_{\mathrm{G}}}(\exp(-X) \cdot \tau, \tau).$$

Thanks to equations (1.1.1) and (1.1.2) the proof is complete. □

Another geometric interpretation of the polar projection, directly in the space X, is proposed by Kassel-Kobayashi in [33, p.151]: in order to go from o to $g \cdot o$, we first travel along the flat subspace[6]

$$\exp(\mathfrak{b}) \cdot o$$

a "distance" $\|b^{o,\tau}(g)\|_{\mathfrak{b}}$, and then we move along the compact K^τ-orbit of the point $\exp(b^{o,\tau}(g)) \cdot o$.

Let us illustrate this in the more concrete case $\mathsf{X} = \mathsf{H}^{p,q-1}$. Let

$$\mathsf{H}^p_\tau := (\tau^{\perp_{p,q}} \oplus o) \cap \mathsf{H}^{p,q-1}.$$

Observe that H^p_τ is a totally geodesic copy of the p-dimensional hyperbolic space H^p inside $\mathsf{H}^{p,q-1}$, because the subspace $\tau^{\perp_{p,q}} \oplus o$ has signature $(p,1)$ for the form $\langle \cdot, \cdot \rangle_{p,q}$. It can be seen that the tangent space at o of H^p_τ naturally identifies with

[6] This flat subspace can be proven to be a maximal flat inside a totally geodesic space-like copy of $\mathsf{X}_{\mathrm{G}^{\tau o}}$ inside X, where $\mathrm{G}^{\tau o} := \mathrm{Fix}(\sigma^\tau \sigma^o)$.

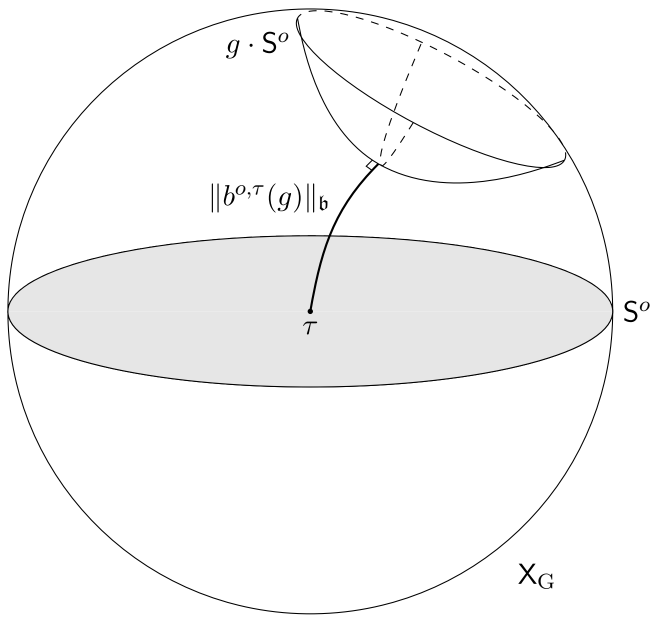


Figure 1.2: Geometric interpretation of polar projection in X_{G}.

$$\mathfrak{p}^\tau \cap \mathfrak{q}^o.$$

Then H^p_τ is space-like and since $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ one has that the geodesic ray $\exp(\mathfrak{b}^+) \cdot o$ is contained in H^p_τ. Let

$$o_g := \exp(b^{o,\tau}(g)) \cdot o.$$

The number $\|b^{o,\tau}(g)\|_{\mathfrak{b}}$ equals then the length of the geodesic segment connecting o with o_g (see Figure 1.3).

1.5 Parabolic subgroups and flag manifolds

In the next chapter we recall the notion of Anosov representation into G. In order to do that, we give a quick reminder on the structure theory of *parabolic subgroups* of G, as well as the notion of *Cartan attractors* and *proximal elements* in G.

Fix a point $\tau \in \mathsf{X}_{\mathrm{G}}$, a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}^\tau$ and a Weyl chamber $\mathfrak{a}^+$ of the system Σ. Denote by $\iota_{\mathfrak{a}^+}$ the *opposition involution* associated to these choices. By definition, if $w_{\mathfrak{a}^+}$ denotes the unique element of the Weyl group that takes the Weyl chamber $-\mathfrak{a}^+$ to $\mathfrak{a}^+$ then

$$\iota_{\mathfrak{a}^+} : \mathfrak{a} \to \mathfrak{a} : \quad \iota_{\mathfrak{a}^+}(X) := -w_{\mathfrak{a}^+} \cdot X.$$

Let Σ^+ be the positive system and $\Delta \subset \Sigma^+$ be the set of simple roots. For any non empty subset $\theta \subset \Delta$ define

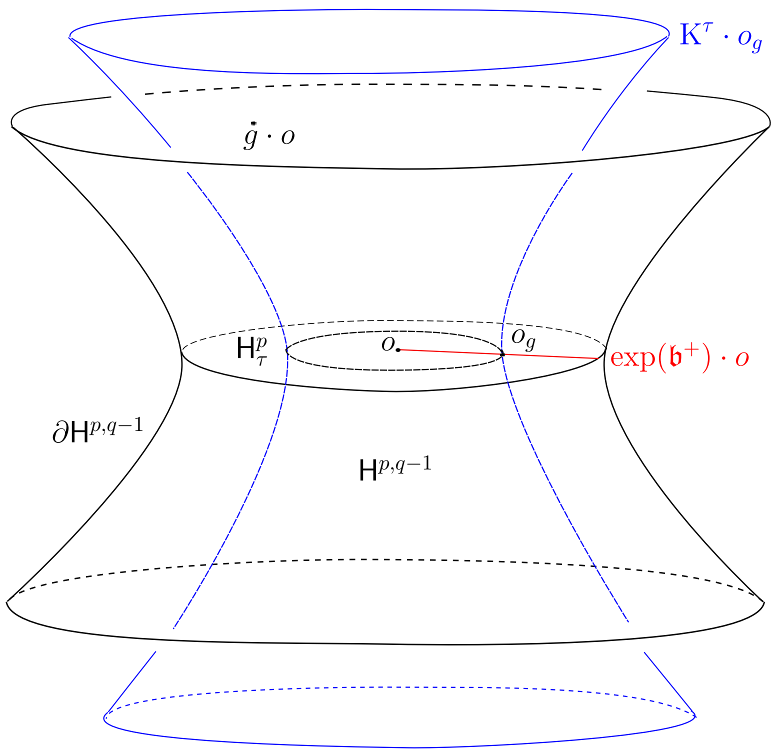


Figure 1.3: Geometric interpretation of polar projection in $\mathsf{H}^{p,q-1}$.

$$\mathfrak{p}_\theta := \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\operatorname{span}(\Delta\setminus\theta)\cap\Sigma^+} \mathfrak{g}_{-\alpha}$$

and similarly set

$$\check{\mathfrak{p}}_\theta := \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_{-\alpha} \oplus \bigoplus_{\alpha\in\operatorname{span}(\Delta\setminus\theta)\cap\Sigma^+} \mathfrak{g}_\alpha.$$

The subgroup of G associated to the Lie algebra $\mathfrak{p}_\theta$ (resp. $\check{\mathfrak{p}}_\theta$) is denoted by P_θ (resp. $\check{\mathrm{P}}_\theta$). The pair $(\check{\mathrm{P}}_\theta, \mathrm{P}_\theta)$ is a pair of *opposite parabolic* subgroups of G. The subgroup P_Δ is a minimal parabolic subgroup of G, sometimes also called a *Borel subgroup* of G, and it is canonically determined by the choice of the Weyl chamber $\mathfrak{a}^+$. Because of this, we will sometimes denote it by $\mathrm{P}_{\mathfrak{a}^+}$.

Define

$$\iota_{\mathfrak{a}^+}(\theta) := \{\alpha \circ \iota_{\mathfrak{a}^+} : \quad \alpha \in \theta\}.$$

The subgroup $\check{\mathrm{P}}_\theta$ is conjugate to $\mathrm{P}_{\iota_{\mathfrak{a}^+}(\theta)}$. The parabolic subgroup P_θ is said to be *self opposite* if $\iota_{\mathfrak{a}^+}(\theta) = \theta$.

A *flag manifold* of G is a manifold of the form $\mathsf{F}_\theta := \mathrm{G}/\mathrm{P}_\theta$, for some $\emptyset \neq \theta \subset \Delta$. The corresponding "opposite" flag manifold $\mathrm{G}/\check{\mathrm{P}}_\theta$ will be denoted by $\check{\mathsf{F}}_\theta$. The action of G on $\check{\mathsf{F}}_\theta \times \mathsf{F}_\theta$ admits a unique open orbit, denoted by $(\check{\mathsf{F}}_\theta \times \mathsf{F}_\theta)^{(2)}$ (or by $\mathsf{F}_\theta^{(2)}$ if P_θ is self opposite). The flag $\check{\xi} \in \check{\mathsf{F}}_\theta$ is said to be *transverse* to the flag $\xi \in \mathsf{F}_\theta$ if the pair $(\check{\xi}, \xi)$ belongs to this open orbit.

Example 1.5.1. We give here two explicit examples of flag manifolds that will be important in this tesis. We also fix some particular terminology and notations in each case.

- When $\mathrm{G} = \mathrm{PSO}(p,q)$, let $\mathrm{P}_1^{p,q}$ be the stabilizer in G of an *isotropic line* in $\mathbb{R}^{p,q}$, i.e. an element $\xi = [\hat{\xi}]$ in the projective space $\mathsf{P}(\mathbb{R}^{p,q})$ for which one has

$$\langle \hat{\xi}, \hat{\xi} \rangle_{p,q} = 0.$$

  It can be seen that $\mathrm{P}_1^{p,q}$ is a parabolic subgroup of G. The associated flag manifold is identified with the set of isotropic lines in $\mathsf{P}(\mathbb{R}^{p,q})$ and it is called the *boundary* of $\mathsf{H}^{p,q-1}$. We denote this flag manifold by $\partial\mathsf{H}^{p,q-1}$. The corresponding opposite flag manifold is identified in this case with the set

$$\{\xi^{\perp_{p,q}} : \quad \xi \in \partial\mathsf{H}^{p,q-1}\}.$$

  A pair $(\xi_1^{\perp_{p,q}}, \xi_2)$ with ξ_1 and ξ_2 in $\partial\mathsf{H}^{p,q-1}$ is transverse if and only if the hyperplane $\xi_1^{\perp_{p,q}}$ is linearly disjoint from the line ξ_2.

- Suppose now that $\mathrm{G} = \mathrm{PSL}(V)$. For $j = 1, \dots, d$, denote by $\mathsf{Gr}_j(V)$ the *Grassmannian* of j-dimensional subspaces of V. Denote by $\mathsf{F}(V)$ the space of complete (or full) flags of V, that is,

$$\mathsf{F}(V) := \{\xi = (\xi^1 \subset \dots \subset \xi^d) : \quad \xi^j \in \mathsf{Gr}_j(V) \text{ for every } j = 1, \dots, d\}.$$

  It can be seen that $\mathsf{F}(V)$ identifies with the flag manifold associated to a minimal parabolic subgroup of G. Two complete flags ξ_1 and ξ_2 are transverse if and only if for every $j = 1, \dots, d-1$ one has that the subspace ξ_1^j is linearly disjoint from ξ_2^{d-j}.

◇

1.5.1 Cartan attractors

Fix a non empty subset $\theta \subset \Delta$. An element g of G is said to have a *gap of index* θ if for every element $\alpha \in \theta$ one has

$$\alpha(a^\tau(g)) > 0.$$

In this case, write

$$g = k \exp(a^\tau(g)) l$$

a Cartan decomposition of g. The *Cartan attractor* of g in F_θ is the flag

$$U_\theta^\tau(g) := k \cdot \xi_\theta \in \mathsf{F}_\theta,$$

where ξ_θ denotes the coset P_θ in the flag manifold F_θ. Since g has a gap of index θ, it does not depend on the particular choice of the Cartan decomposition of g (c.f. [35, Chapter VII]).

Note that g has a gap of index θ if and only if g^{-1} has a gap of index $\iota_{\mathfrak{a}^+}(\theta)$. In this case we denote

$$S_\theta^\tau(g) := U^\tau_{\iota_{\mathfrak{a}^+}(\theta)}(g^{-1}) \in \check{\mathsf{F}}_\theta,$$

and call this flag the *Cartan repellor* of g in $\check{\mathsf{F}}_\theta$.

The following remark is classical (see for instance Stecker [63, Lemma 3.1.3]).

Remark 1.5.2. Let $\{g_n\}_{n\geq 0}$ be a sequence in G such that

$$\min_{\alpha\in\theta} \alpha(a^\tau(g_n)) \to \infty$$

as $n \to \infty$. Assume further that one has

$$U_\theta^\tau(g_n) \to \xi \text{ and } S_\theta^\tau(g_n) \to \check{\xi}$$

for some flags $\xi \in \mathsf{F}_\theta$ and $\check{\xi} \in \check{\mathsf{F}}_\theta$. Then given any compact set $C \subset \mathsf{F}_\theta$ whose elements are transverse to the flag $\check{\xi}$, the sequence

$$g_n|_C : C \to \mathsf{F}_\theta$$

converges uniformly to the constant function ξ as $n \to \infty$.

$\diamond$

1.5.2 Proximality

Given a flag $\check{\xi}$ in $\check{\mathsf{F}}_\theta$ we denote by

$$\mathtt{V}(\check{\xi}) := \{\xi \in \mathsf{F}_\theta : \quad \xi \text{ is not tranverse to } \check{\xi}\}$$

the *annihilator* of ξ. Note that $\mathtt{V}(\check{\xi})$ is a compact subset of F_θ. An element g in G is said to be *proximal* in F_θ (or F_θ-*proximal*, or P_θ-*proximal*) if it has two fixed points ξ and $\check{\xi}$ in F_θ and $\check{\mathsf{F}}_\theta$ respectively, with $\xi \notin \mathtt{V}(\check{\xi})$, and such that for every $\xi_1 \notin \mathtt{V}(\check{\xi})$, one has

$$g^n \cdot \xi_1 \to \xi$$

as $n \to \infty$. In this case, the fixed points ξ and $\check{\xi}$ are uniquely determined by g and are denoted respectively by $g_+ \in \mathsf{F}_\theta$ and $g_- \in \check{\mathsf{F}}_\theta$.

Fix a continuous distance $d(\cdot,\cdot)$ on F_θ, a positive ε and flags $\xi \in \mathsf{F}_\theta$ and $\check{\xi} \in \check{\mathsf{F}}_\theta$. Let

$$b_\varepsilon(\xi) := \{\xi_1 \in \mathsf{F}_\theta : \quad d(\xi_1, \xi) \leq \varepsilon\}$$

and

$$B_\varepsilon(\check{\xi}) := \{\xi_1 \in \mathsf{F}_\theta : \quad d(\xi_1, \mathtt{V}(\check{\xi})) \geq \varepsilon\},$$

where $d(\xi_1, \mathtt{V}(\check{\xi}))$ denotes the minimal distance between the flag ξ_1 and points in the annihilator $\mathtt{V}(\check{\xi})$.

An important notion is the following quantified version of proximality. Fix numbers $0 < \varepsilon \leq r$ and an element $g \in \mathrm{G}$ which is F_θ-proximal. We say that g is (r,ε)-*proximal* (in F_θ) if one has

$$d(g_+, \mathtt{V}(g_-)) \geq 2r$$

and[7]

$$g \cdot B_\varepsilon(g_-) \subset b_\varepsilon(g_+).$$

1.6 Iwasawa decomposition

We finish this chapter recalling the *Iwasawa decomposition* of G. This decomposition plays a central role in the study of the Cartan projection of G (c.f. Quint [56]).

Keep the notation from the previous subsection and consider the Lie algebras

[7]Sometimes one also requires that the restriction $g|_{B_\varepsilon(g_-)}$ is ε-contracting. We do not need this assumption.

$$\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \text{ and } \check{\mathfrak{n}} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}.$$

Let N and $\check{\mathrm{N}}$ be the corresponding connected Lie subgroups of G. The Iwasawa decomposition theorem states that the map

$$\mathrm{K}^\tau \times \mathfrak{a} \times \mathrm{N} \to \mathrm{G} : \quad (k, X, n) \mapsto k \exp(X) n \tag{1.6.1}$$

is a diffeomorphism (see [35, Chapter VI]).

1.6.1 Busemann cocycle of G

Let $\xi_{\mathfrak{a}^+}$ be the coset $\mathrm{P}_{\mathfrak{a}^+}$ in the flag manifold $\mathsf{F}_{\mathfrak{a}^+} = \mathrm{G}/\mathrm{P}_{\mathfrak{a}^+}$. Quint [56] introduces the[8] *Busemann cocycle of* G

$$\beta^\tau : \mathrm{G} \times \mathsf{F}_{\mathfrak{a}^+} \to \mathfrak{b}$$

which is defined by the equality

$$gk = l \exp(\beta^\tau(g, \xi)) n$$

where $k, l \in \mathrm{K}^\tau$, $n \in \mathrm{N}$ and $k \cdot \xi_{\mathfrak{a}^+} = \xi$. Note that for every g_1 and g_2 in G, and every $\xi \in \mathsf{F}_{\mathfrak{a}^+}$ one has

$$\beta^\tau(g_1 g_2, \xi) = \beta^\tau(g_1, g_2 \cdot \xi) + \beta^\tau(g_2, \xi).$$

In this thesis we call this cocycle the τ-*Busemann cocycle* of G.

[8] Sometimes also called the *Iwasawa cocycle* of G.

Chapter 2

Anosov representations

Anosov representations were introduced by Labourie [37] for surface groups and extended by Guichard-Wienhard [26] to word hyperbolic groups. In Section 2.1 we recall this notion and some well known facts concerning (r, ε)-proximality of elements in the image of such a representation. We also discuss some examples. In Section 2.2 we recall the definition (and main dynamical properties) of the *geodesic flow* of a projective Anosov representation, introduced by Bridgeman-Canary-Labourie-Sambarino [12]. Section 2.3 is devoted to reminders on the work of Sambarino [60], which is an important tool that we will use in Parts II and III to prove our counting results.

2.1 Definition and first properties

2.1.1 Definition

A lot of work has been done in order to simplify the Labourie's original definition of Anosov representations. Here we follow mainly the work of Bochi-Potrie-Sambarino [7], Guichard-Guéritaud-Kassel-Wienhard [25] and Kapovich-Leeb-Porti [29].

Let Γ be a finitely generated group. Consider a finite symmetric generating set S of Γ and take $|\cdot|_\Gamma$ to be the associated word length: for γ in Γ, it is the minimum number required to write γ as a product of elements of S[1].

Let G be a connected semisimple Lie group with finite center and no compact factors and fix a point τ in X_G. Let $\mathfrak{a}^+$ be a closed Weyl chamber of the system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$, for some Cartan subspace $\mathfrak{a} \subset \mathfrak{p}^\tau$. Fix a non empty subset $\theta \subset \Delta$.

[1]This number depends on the choice of S. However, the set S will be fixed from now on hence we do not emphasize the dependence on this choice in the notation.

Let $\rho : \Gamma \to \mathrm{G}$ be a representation. We say that ρ is *θ-Anosov* (or *P_θ-Anosov*) if there exist positive constants c and c' such that for all $\gamma \in \Gamma$ and all $\alpha \in \theta$ one has

$$\alpha(a^\tau(\rho\gamma)) \geq c|\gamma|_\Gamma - c'. \tag{2.1.1}$$

When θ coincides with the set of simple roots Δ, the representation ρ is said to be *Borel-Anosov*.

Remark 2.1.1. Every θ-Anosov representation is a quasi isometric embedding. In particular, it is discrete and has finite kernel.

◇

By Kapovich-Leeb-Porti [31, Theorem 1.4] (see also [7, Section 3]), condition (2.1.1) implies that Γ is word hyperbolic[2]. In this thesis we assume that Γ is non elementary. Let $\partial_\infty\Gamma$ be the Gromov boundary of Γ and Γ_{H} be the set of infinite order elements in Γ. Every γ in Γ_{H} has exactly two fixed points in $\partial_\infty\Gamma$: the attractive one denoted by γ_+ and the repelling one denoted by γ_-. The dynamics of γ on $\partial_\infty\Gamma$ is of type "north-south".

Fix a θ-Anosov representation $\rho : \Gamma \to \mathrm{G}$. By [7, 25, 29] there exist continuous equivariant maps

$$\xi_{\rho,\theta} : \partial_\infty\Gamma \to \mathsf{F}_\theta \text{ and } \check{\xi}_{\rho,\theta} : \partial_\infty\Gamma \to \check{\mathsf{F}}_\theta$$

which are *transverse*, i.e. for every $x \neq y$ in $\partial_\infty\Gamma$ one has

$$(\check{\xi}_{\rho,\theta}(x), \xi_{\rho,\theta}(y)) \in (\check{\mathsf{F}}_\theta \times \mathsf{F}_\theta)^{(2)}. \tag{2.1.2}$$

Moreover, these maps are *dynamics-preserving*. This means that for every γ in Γ_{H} the element $\xi_{\rho,\theta}(\gamma_+)$ (resp. $\check{\xi}_{\rho,\theta}(\gamma_+)$) is an attractive fixed point of $\rho\gamma$ acting on F_θ (resp. $\check{\mathsf{F}}_\theta$). It follows that $\rho\gamma$ is proximal in F_θ with

$$\xi_{\rho,\theta}(\gamma_+) = (\rho\gamma)_+ \text{ and } \check{\xi}_{\rho,\theta}(\gamma_-) = (\rho\gamma)_-.$$

In particular, both $\xi_{\rho,\theta}$ and $\check{\xi}_{\rho,\theta}$ are injective and uniquely determined by ρ. These are called the *limit maps* of ρ. Note that when P_θ is self opposite both limit maps coincide.

Labourie's (and Guichard-Wienhard's) definition of Anosov representations involves the construction of certain flow space associated to ρ which satisfies a "uniform contraction/dilation property", the contracting and dilating "directions" being directly related to the limit maps (see [37, 26] for precisions). Standard techniques coming from hyperbolic dynamics give then the following (for a proof see [26, Theorem 5.13] and [12, Theorem 6.1]).

---

[2] We refer the reader to the book of Ghys-de la Harpe [22] for definitions and standard facts on word hyperbolic groups.

Proposition 2.1.2. *The space of θ-Anosov representations is open in the space of homomorphisms from Γ to* G*. The limit maps of an Anosov representation are Hölder continuous and vary continuously with the representation.*

□

We denote by $\mathrm{L}_{\rho,\theta}$ and $\check{\mathrm{L}}_{\rho,\theta}$ the respective images of $\xi_{\rho,\theta}$ and $\check{\xi}_{\rho,\theta}$. These are called the *limit sets* of $\rho(\Gamma)$ and admit the following characterization (see [25, Theorem 5.3] or [7, Subsection 3.4] for a proof).

Proposition 2.1.3. *Let $\rho : \Gamma \to \mathrm{G}$ be a θ-Anosov representation. Then $\mathrm{L}_{\rho,\theta}$ (resp. $\check{\mathrm{L}}_{\rho,\theta}$) coincides with the set of accumulation points of sequences of the form $\{U_\theta^\tau(\rho\gamma_n)\}_n$ (resp. $\{S_\theta^\tau(\rho\gamma_n)\}_n$), where $\gamma_n \to \infty$. Moreover, given (continuous) distances $d(\cdot,\cdot)$ and $\check{d}(\cdot,\cdot)$ on F_θ and $\check{\mathsf{F}}_\theta$ respectively and any positive ε, one has*

$$d\left(U_\theta^\tau(\rho\gamma),(\rho\gamma)_+\right) < \varepsilon \text{ and } \check{d}(S_\theta^\tau(\rho\gamma),(\rho\gamma)_-) < \varepsilon$$

for every $\gamma \in \Gamma_{\mathrm{H}}$ with $|\gamma|_\Gamma$ large enough.

□

2.1.2 Proximality properties

Anosov representations have strong proximality properties (see Guichard-Wienhard [26, Subsection 5.2]). We now establish one of them that will be useful in Parts II and III of this thesis: they will provide the correct framework to estimate the geometric quantities involved in our counting functions.

Let $\rho : \Gamma \to \mathrm{G}$ be a θ-Anosov representation and fix a continuous distance $d(\cdot,\cdot)$ on F_θ.

Lemma 2.1.4 (c.f. Sambarino [60, Lemma 5.7])**.** *Fix real numbers $0 < \varepsilon \leq r$. Then there exists a positive L such that for every $\gamma \in \Gamma_{\mathrm{H}}$ satisfying $|\gamma|_\Gamma > L$ and*

$$d((\rho\gamma)_+, \mathtt{V}((\rho\gamma)_-)) \geq 2r,$$

one has that $\rho\gamma$ is (r,ε)-proximal in F_θ.

Proof. Consider a sequence $\gamma_n \to \infty$ in Γ_{H} such that

$$d((\rho\gamma_n)_+, \mathtt{V}((\rho\gamma_n)_-)) \geq 2r$$

for all n. By Proposition 2.1.3, for every n large enough the following holds

$$b_{\frac{\varepsilon}{2}}(U_\theta^\tau(\rho\gamma_n)) \subset b_\varepsilon((\rho\gamma_n)_+)$$

and

$$B_\varepsilon((\rho\gamma_n)_-) \subset B_{\frac{\varepsilon}{2}}(S_\theta^\tau(\rho\gamma_n)).$$

By Remark 1.5.2 and equation (2.1.1) the condition

$$\rho\gamma_n \cdot B_\varepsilon((\rho\gamma_n)_-) \subset b_\varepsilon((\rho\gamma_n)_+)$$

is satisfied for sufficiently large n.

□

2.1.3 Examples

The class of Anosov representations is very rich an provides a unified framework for examples coming from different constructions. We will now discuss few of these examples for the Lie groups $\mathrm{G} = \mathrm{PSL}(V)$ and $\mathrm{G} = \mathrm{PSO}(p,q)$ (see the surveys of Kassel [32] or Wienhard [64] and references therein for other examples).

Remark 2.1.5. It is standard to use the following special terminology. A representation

$$\rho : \Gamma \to \mathrm{PSL}(V)$$

which is Anosov with respect to the parabolic subgroup of $\mathrm{PSL}(V)$ stabilizing a point in $\mathsf{P}(V)$ will be called *projective Anosov*. Note that a representation

$$\rho : \Gamma \to \mathrm{PSO}(p,q)$$

is $\mathrm{P}_1^{p,q}$-Anosov if and only if the representation

$$\Gamma \xrightarrow{\rho} \mathrm{PSO}(p,q) \hookrightarrow \mathrm{PSL}(\mathbb{R}^{p,q})$$

is projective Anosov.

◇

Some examples for $\mathrm{G} = \mathrm{PSL}(V)$

In Part III of this thesis we will be interested in Borel-Anosov representations into $\mathrm{PSL}(V)$, hence we focus here on examples of these type of representations.

Example 2.1.6 (Schottky subgroups). The simplest way of constructing Borel-Anosov subgroups of $\mathrm{PSL}(V)$ is to use a "ping-pong construction": fix a family

$$\{\xi_1, \dots, \xi_s, \check{\xi}_1, \dots, \check{\xi}_s\}$$

of pairwise transverse elements in $\mathsf{F}(V)$. Let τ be an element in $\mathsf{X}_{\mathrm{PSL}(V)}$ and a positive ε such that, for every $i, j = 1, \dots, s$ one has[3]

$$B_\varepsilon(\check{\xi}_i) \subset b_\varepsilon(\xi_j) \text{ and } B_\varepsilon(\xi_i) \subset b_\varepsilon(\check{\xi}_j).$$

For each $i = 1, \dots, s$ let g_i be an element in $\mathrm{PSL}(V)$ satisfying

$$U^\tau(g_i) = \xi_i \text{ and } S^\tau(g_i) = \check{\xi}_i,$$

where $U^\tau(g_i) := U^\tau_\Delta(g_i)$ and $S^\tau(g_i) := S^\tau_\Delta(g_i)$. Further suppose that

$$g_i \cdot B_\varepsilon(\check{\xi}_i) \subset b_\varepsilon(\xi_i) \text{ and } g_i^{-1} \cdot B_\varepsilon(\xi_i) \subset b_\varepsilon(\check{\xi}_i).$$

Let Γ be the (free) subgroup of $\mathrm{PSL}(V)$ generated by the elements $g_1, \dots, g_s$. The inclusion representation

$$\Gamma \hookrightarrow \mathrm{PSL}(V)$$

is Borel-Anosov (c.f. [7, Lemma A.7]).

Similar constructions work to produce θ-Anosov (free) subgroups of $\mathrm{PSL}(V)$, for any parabolic subgroup P_θ of $\mathrm{PSL}(V)$. All these subgroups will be called of *Schottky type*.

◇

Example 2.1.7 (Teichmüller representations). Let $g \geq 2$ be an integer and Γ_g be the fundamental group of a closed orientable surface of genus g. By the Švarc-Milnor Lemma (see [22, Proposition 19 of Ch. 3]), a representation

$$\rho : \Gamma_g \to \mathrm{PSL}_2(\mathbb{R})$$

is (Borel-)Anosov if and only if it is faithful (i.e. injective) and discrete. These representations will be called *Teichmüller representations*.

◇

Let

$$\Lambda_d^{\mathrm{irr}} : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}(V)$$

be the representation induced by the (unique up to conjugation) irreducible action of $\mathrm{SL}_2(\mathbb{R})$ on V. For any $\mu > 1$, the image of the diagonal matrix

$$a_\mu := \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$$

[3]As explained in Appendix B, the choice of τ induces a continuous distance $d(\cdot,\cdot)$ on $\mathsf{F}(V)$.

is a diagonalizable element of $\mathrm{SL}(V)$ with eigenvalues

$$\mu^{d-1}, \mu^{d-3}, \ldots, \mu^{3-d}, \mu^{1-d}. \tag{2.1.3}$$

Example 2.1.8 (Hitchin representations). Let

$$\rho_0 : \Gamma_g \to \mathrm{PSL}(V)$$

be the representation induced by the composition of (a lifted to $\mathrm{SL}_2(\mathbb{R})$ of) a Teichmüller representation with Λ_d^{irr}. Then ρ_0 is Borel-Anosov, essentially because of the previous consideration on the eigenvalues of $\Lambda_d^{\mathrm{irr}}(a_\mu)$ for $\mu > 1$. Labourie's Theorem [37] states that any deformation of ρ_0 is Borel-Anosov. Representations obtained in this way are called *Hitchin representations*.

◇

Example 2.1.9 (Deformations of reducible representations). Let

$$\Lambda : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}(V)$$

be a representation. We look at a representation

$$\rho_0 : \Gamma_g \to \mathrm{PSL}(V)$$

obtained from Λ and a Teichmüller representation as in the previous example. Since Λ can be written as a direct sum of irreducible representations, it induces some parabolic subgroups of $\mathrm{PSL}(V)$ for which ρ_0 is Anosov. Note that if we assume that Λ is reducible then ρ_0 will be Borel-Anosov if and only if Λ splits as direct sum of two irreducible representations of dimensions d_1 and d_2, for some integers d_1 and d_2 which are different modulo 2. By Proposition 2.1.2 small deformations of such a ρ_0 are still Borel-Anosov. We call these representations of *Barbot type* (c.f. Barbot [2]).

◇

Some examples for $\mathrm{G} = \mathrm{PSO}(p, q)$

In Part II we will focus on $\mathrm{P}_1^{p,q}$-Anosov representations into $\mathrm{PSO}(p, q)$, where $\mathrm{P}_1^{p,q}$ is the stabilizer in $\mathrm{PSO}(p, q)$ of an isotropic line in $\mathsf{P}(\mathbb{R}^{p,q})$. As we now discuss, the class of $\mathrm{P}_1^{p,q}$-Anosov representations is rich enough to make its study worthwhile.

Example 2.1.10.

- Fix an integer m such that $p \geq m \geq 2$ and consider a representation

$$\Lambda : \mathrm{SO}(m, 1) \to \mathrm{PSO}(p, q)$$

induced by the choice of a $(m+1)$-dimensional subspace of $\mathbb{R}^{p,q}$ on which the quadratic form $\langle\cdot,\cdot\rangle_{p,q}$ has signature $(m,1)$. Let Γ be the fundamental group of a convex co-compact hyperbolic manifold of dimension m and $\iota_0:\Gamma\to\mathrm{SO}(m,1)$ be the holonomy representation. The representation

$$\rho_0 := \Lambda\circ\iota_0$$

is $\mathrm{P}_1^{p,q}$-Anosov, because ι_0 is $\mathrm{P}_1^{m,1}$-Anosov and Λ is proximal.

- The previous example generalizes to a class of representations studied by Danciger-Guéritaud-Kassel in [17, 18] and called $\mathsf{H}^{p,q-1}$-*convex co-compact*. Very briefly, these are inclusion representations induced by the choice of an infinite discrete subgroup $\Gamma<\mathrm{PSO}(p,q)$ which preserves some properly convex non empty open set $\Omega\subset\mathsf{P}(\mathbb{R}^{p,q})$ whose boundary is strictly convex and of class C^1. One also requires that Γ preserves some "distinguished" non empty convex subset of Ω on which the action is co-compact (see [17, Definitions 1.1 & 1.24 and Theorem 1.25] for precisions). If $\Gamma<\mathrm{PSO}(p,q)$ is an $\mathsf{H}^{p,q-1}$-convex co-compact group, the inclusion representation $\Gamma\hookrightarrow\mathrm{PSO}(p,q)$ is $\mathrm{P}_1^{p,q}$-Anosov (see [17, Theorem 1.25]).

- By a Schottky type construction one can find $\mathrm{P}_1^{p,q}$-Anosov representations into $\mathrm{PSO}(p,q)$ whose image is not $\mathsf{H}^{p,q-1}$-convex co-compact (c.f. [18, Examples 5.2 & 5.3]). However, if $\partial_\infty\Gamma$ is connected every $\mathrm{P}_1^{p,q}$-Anosov representation into $\mathrm{PSO}(p,q)$ is $\mathsf{H}^{p,q-1}$-convex co-compact (or $\mathsf{H}^{q,p-1}$-convex co-compact) as shown in [18, Theorem 1.7].

$\diamond$

2.2 The geodesic flow of a projective Anosov representation

For the remainder of this chapter we denote $\mathrm{G}:=\mathrm{PSL}(V)$ and we fix a projective Anosov representation

$$\rho:\Gamma\to\mathrm{G}.$$

The main goal of this section is to recall the definition of the *geodesic flow* of ρ (introduced in [12]) and to discuss some of its dynamical properties. Notably, we are interested in some explicit descriptions of its topological entropy and its probability of maximal entropy (Fact 2.2.4 below). The key tool to obtain these descriptions is the fact that the geodesic flow of ρ is a (transitive) *metric Anosov flow* (see Pollicott [51] or Appendix A for reminders on this notion). Some specific reparametrizations of the geodesic flow of ρ will be used in Parts II and III to obtain our counting results.

Note that we have two limit maps in this case, one with image in projective space $\mathsf{P}(V)$ and the other one with image in the Grassmannian $\mathsf{Gr}_{d-1}(V)$. To make formulas more readable, we denote this maps by

$$\xi : \partial_\infty\Gamma \to \mathsf{P}(V) \text{ and } \eta : \partial_\infty\Gamma \to \mathsf{Gr}_{d-1}(V).$$

We also use the special notation L^1_ρ for the image of ξ in this case.

2.2.1 Definition and the metric Anosov property

The standard reference for this subsection is the work of Bridgeman-Canary-Labourie-Sambarino [12]. Let

$$\partial^2_\infty\Gamma := \{(x,y) \in (\partial_\infty\Gamma)^2 : \quad x \neq y\}.$$

For a point $(x,y) \in \partial^2_\infty\Gamma$ define[4]

$$\mathrm{M}(x,y) := \{(\vartheta, v) \in \eta(x) \times \xi(y) : \quad \vartheta(v) = 1\}/\sim,$$

where $(\vartheta, v) \sim (-\vartheta, -v)$. Consider the line bundle over $\partial^2_\infty\Gamma$ defined by

$$F_\rho := \{(x,y,\vartheta,v) : \quad (x,y) \in \partial^2_\infty\Gamma \text{ and } (\vartheta,v) \in \mathrm{M}(x,y)\}.$$

Fact 2.2.1 (see [12, Sections 4 and 5])**.** The following holds:

- The action of Γ on F_ρ induced by ρ is proper and co-compact. The quotient space is denoted by $\mathrm{U}_\rho\Gamma$.

- The flow ϕ_t on F_ρ defined by

$$\phi_t(x,y,\vartheta,v) := (x,y,e^{-t}\vartheta, e^t v)$$

  descends to a flow on $\mathrm{U}_\rho\Gamma$, still denoted by ϕ_t, and called the *geodesic flow* of ρ. The geodesic flow of ρ is conjugate, by a Hölder homeomorphism, to a Hölder reparametrization[5] of the Gromov geodesic flow of Γ (see Mineyev [44] for a definition of the latter).

- Periodic orbits of ϕ_t are in one to one correspondence with conjugacy classes of *primitive* elements γ in Γ, that is, elements that cannot be written as a positive power of another element in Γ. The period of the periodic orbit corresponding to the conjugacy class $[\gamma]$ of a primitive element γ is $\lambda_1(\rho\gamma)$.

[4]Here we use the natural identification $\mathsf{Gr}_{d-1}(V) \cong \mathsf{P}(V^*)$ (see Subsection B.1.1).

[5]In Appendix A we recall the notion of *Hölder reparametrization* of a continuous flow.

- The geodesic flow ϕ_t is a transitive metric Anosov flow (see Appendix A for reminders on this notion). Explicitly, given a point $Z_0 = (x_0, y_0, \vartheta_0, v_0)$ in $\mathrm{U}_\rho\Gamma$ the strong stable and strong unstable leaves through Z_0 are given by:

$$\mathscr{W}^{ss}(Z_0) = \{(x, y_0, \vartheta, v_0) \in \mathrm{U}_\rho\Gamma : \quad \vartheta \in \eta(x) \text{ and } \vartheta(v_0) = 1\}$$

and

$$\mathscr{W}^{uu}(Z_0) = \{(x_0, y, \vartheta_0, v) \in \mathrm{U}_\rho\Gamma : \quad v \in \xi(y) \text{ and } \vartheta_0(v) = 1\}.$$

The central stable and central unstable leaves are given by:

$$\mathscr{W}^{cs}(Z_0) = \{(x, y_0, \vartheta, v) \in \mathrm{U}_\rho\Gamma : \quad \vartheta \in \eta(x),\ v \in \xi(y_0) \text{ and } \vartheta(v) = 1\}$$

and

$$\mathscr{W}^{cu}(Z_0) = \{(x_0, y, \vartheta, v) \in \mathrm{U}_\rho\Gamma : \ v \in \xi(y),\ \vartheta \in \eta(x_0) \text{ and } \vartheta(v) = 1\}.$$

◇

2.2.2 Entropy and distribution of periodic orbits

The following proposition, interesting in its own right, provides the correct framework on which our counting results are based. The proof is inspired by Benoist's work [5].

Proposition 2.2.2. *Let $\rho : \Gamma \to \mathrm{PSL}(V)$ be a projective Anosov representation. Then the subset*

$$\{\lambda_1(\rho\gamma) : \quad \gamma \in \Gamma\} \subset \mathbb{R}$$

spans a dense subgroup of $\mathbb{R}$.

Proof. Suppose by contradiction that there exists a constant $a > 0$ such that the group spanned by the set $\{\lambda_1(\rho\gamma)\}_{\gamma\in\Gamma}$ is contained in $a\mathbb{Z}$.

Denote by

$$\partial^4_\infty\Gamma := \{(x_1, x_2, x_3, x_4) \in (\partial_\infty\Gamma)^4 : \quad (x_i, x_j) \in \partial^2_\infty\Gamma \text{ for all } i \neq j\}.$$

Since $\{(\gamma_-, \gamma_+)\}_{\gamma\in\Gamma_{\mathrm{H}}}$ is dense in $\partial^2_\infty\Gamma$ (see Gromov [24, Corollary 8.2.G]), Benoist's Theorem B.1.5 implies that

$$\{\mathbb{B}^1(\eta(x'), \xi(y'), \eta(x), \xi(y)) : \quad (x', y', x, y) \in \partial^4_\infty\Gamma\} \subset a\mathbb{Z}, \tag{2.2.1}$$

where $\mathbb{B}^1$ is the projective cross-ratio (see Subsection B.1.1 of Appendix B for definitions).

Fix three different points x', y' and y in $\partial_\infty\Gamma$. Transversality condition (2.1.2) implies that the intersection

$$\eta(x) \cap (\xi(y) \oplus \xi(y'))$$

is one dimensional for every $x \notin \{y, y'\}$. Equation (2.2.1) implies then that for every $x \in \partial_\infty \Gamma$ such that $(x', y', x, y) \in \partial_\infty^4 \Gamma$ there exists a neighbourhood V of x and a point $\xi_{x,y,y'}$ in the projective line $\xi(y) \oplus \xi(y')$ such that

$$\eta(\tilde{x}) \cap (\xi(y) \oplus \xi(y')) = \{\xi_{x,y,y'}\} \tag{2.2.2}$$

holds for every $\tilde{x} \in V$.

Claim 2.2.3. Assume that (2.2.2) holds. Then the limit set L_ρ^1 is not contained in $\xi(y) \oplus \xi(y')$.

Proof of Claim 2.2.3. Suppose by contradiction that $\mathrm{L}_\rho^1 = \xi(y) \oplus \xi(y')$. Then one has

$$\xi(\tilde{x}) \in \eta(\tilde{x}) \cap \mathrm{L}_\rho^1 = \eta(\tilde{x}) \cap (\xi(y) \oplus \xi(y'))$$

and equation (2.2.2) implies $\xi(\tilde{x}) = \xi_{x,y,y'}$ for every $\tilde{x} \in V$. This contradicts the injectivity of ξ.

□

Because of Claim 2.2.3 we can take y'' in $\partial_\infty \Gamma$ such that $\xi(y'')$ does not belong to $\xi(y) \oplus \xi(y')$. We can assume further that $y'' \neq x'$.

By equation (2.2.1) we have again that for every $x \notin \{x', y, y', y''\}$ there exists a neighbourhood V of x and a point $\xi_{x,y,y''}$ in the projective line $\xi(y) \oplus \xi(y'')$ such that

$$\eta(\tilde{x}) \cap (\xi(y) \oplus \xi(y'')) = \{\xi_{x,y,y''}\}$$

holds for every $\tilde{x} \in V$.

As in Claim 2.2.3 we conclude that the limit set L_ρ^1 cannot be contained in $\xi(y) \oplus \xi(y') \oplus \xi(y'')$ and now an inductive argument yields the desired contradiction.

□

In dynamical language, Proposition 2.2.2 states that the geodesic flow ϕ_t is *topologically weakly-mixing*. On the other hand, recall that it is a transitive metric Anosov flow. The work of Pollicott [51] implies then that it admits a *Markov coding*. Further this coding is *strong* (see [12, 16]). All these facts imply the following (the reader is referred to Appendix A for precisions).

Fact 2.2.4 (c.f. Fact A.1.1)**.** The following holds:

- The topological entropy h_ρ^1 of ϕ_t is positive and finite. It is called the *entropy* of ρ and it is given by

$$h_\rho^1 = \lim_{t\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \quad \gamma \text{ is primitive and } \lambda_1(\rho\gamma) \leq t\}}{t}.$$

- There exists a unique probability m_ρ of maximal entropy for ϕ_t, called the *Bowen-Margulis probability* of ρ. Periodic orbits become equidistributed with respect to this measure.

◇

2.3 Hopf coordinates

Let M be a closed negatively curved manifold and $\pi_1(M)$ be its fundamental group (for any basepoint in M). In [60] Sambarino explains how Hölder reparametrizations of the geodesic flow of M correspond to Hölder cocycle over $\partial_\infty\pi_1(M)$. Further, he uses Patterson-Sullivan theory to describe the probability of maximal entropy of this reparametrization. Whenever an Anosov representation of $\partial_\infty\pi_1(M)$ is given, the author explains how these results and Roblin's method [58] combine to obtain some counting results in this higher rank setting. We now recall part of these results and explain a way to adapt them to cocycles over a general word hyperbolic group admitting an Anosov representation: the geodesic flow of M is replaced by the geodesic flow of ρ. Since in Parts II and III we will be interested in some specific reparametrizations (and not just "any" reparametrization) of the geodesic flow, we will focus here in concrete examples of Hölder cocycles over Γ associated to ρ, namely, "Busemann cocycles".

2.3.1 Hölder cocycles and Livšic cohomology

We first fix some terminology that will remain valid throughout the thesis.

Hölder cocycles

A *Hölder cocycle* over $\partial_\infty\Gamma$ is a function

$$c : \Gamma \times \partial_\infty\Gamma \to \mathbb{R}$$

satisfying that for every γ_0, γ_1 in Γ and x in $\partial_\infty\Gamma$ one has

$$c(\gamma_0\gamma_1, x) = c(\gamma_0, \gamma_1 \cdot x) + c(\gamma_1, x),$$

and such that the map $c(\gamma_0, \cdot)$ is Hölder continuous (with the same exponent for every γ_0). The *period* of an element $\gamma \in \Gamma_{\mathrm{H}}$ for such a cocycle is defined by the equality

$$p_c(\gamma) := c(\gamma, \gamma_+).$$

Note that the period of γ only depends on the conjugacy class of γ. Whenever c has strictly positive periods, we let

$$h_c := \limsup_{t\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] : \quad p_c(\gamma) \leq t\}}{t} \in [0, \infty]$$

be the *entropy* of c.

Patterson-Sullivan probabilities

Let $\delta \geq 0$ be a real number. We say that a probability measure μ_c on $\partial_\infty \Gamma$ is a *Patterson-Sullivan probability of dimension* δ for a Hölder cocycle c if the equality[6]

$$\frac{d\gamma_* \mu_c}{d\mu_c}(x) = e^{-\delta c(\gamma^{-1}, x)} \tag{2.3.1}$$

is satisfied for every $\gamma \in \Gamma$. Note that if such a probability exists, the number δ must be strictly positive[7]. Whenever Γ is the fundamental group of a closed negatively curved manifold, a consequence of Ledrappier's work [38] is that if a probability satisfying equality (2.3.1) exists, then the entropy h_c must be strictly positive and finite and further it coincides with δ (in Subsection 2.3.2 below we give a hint of why this is still true in our setting, in a particular case).

Translation flow

Let $\tilde{\mathrm{U}\Gamma} := \partial^2_\infty \Gamma \times \mathbb{R}$ and consider the action of Γ on $\tilde{\mathrm{U}\Gamma}$ given by

$$\gamma \cdot (x, y, s) := (\gamma \cdot x, \gamma \cdot y, s - c(\gamma, y)). \tag{2.3.2}$$

The quotient space will be denoted by $\mathrm{U}_c\Gamma$. The *translation flow* on $\tilde{\mathrm{U}\Gamma}$ defined by

$$\psi_t(x, y, s) := (x, y, s - t) \tag{2.3.3}$$

descends to a flow on $\mathrm{U}_c\Gamma$, still denoted by ψ_t, and still called the *translation flow.*

One can see that Γ-invariant measures on $\partial^2_\infty\Gamma$ are in one to one correspondence with ψ_t-invariant measures on $\mathrm{U}_c\Gamma$. Explicitly, if ν is a Γ-invariant measure on $\partial^2_\infty\Gamma$ the corresponding ψ_t-invariant measure on $\mathrm{U}_c\Gamma$ is

$$m_\nu := \nu \otimes dt.$$

One way of producing Γ-invariant measures on $\partial^2_\infty\Gamma$ is the following. A Hölder cocycle $\bar{c}$ is said to be *dual* to c if the equality

[6] Recall that if $f : X \to Y$ is a map and m is a measure on X then $f_*(m)$ denotes the measure on Y defined by $A \mapsto m(f^{-1}(A))$.

[7] Otherwise μ_c would be a Γ-invariant probability on $\partial_\infty\Gamma$. One can see that Γ-invariant finite measures on $\partial_\infty\Gamma$ do not exist when Γ is non elementary.

$$p_{\overline{c}}(\gamma) = p_c(\gamma^{-1})$$

holds for every $\gamma \in \Gamma_{\mathrm{H}}$. Note that dual cocycles must have the same entropy. A *Gromov product* for the pair $(\overline{c}, c)$ is a map

$$[\cdot,\cdot]_{(\overline{c},c)} : \partial^2_\infty \Gamma \to \mathbb{R}$$

such that for every $\gamma \in \Gamma$ and every $(x, y) \in \partial^2_\infty \Gamma$ one has

$$[\gamma \cdot x, \gamma \cdot y]_{(\overline{c},c)} - [x, y]_{(\overline{c},c)} = -(\overline{c}(\gamma, x) + c(\gamma, y)).$$

If μ_c (resp. $\mu_{\overline{c}}$) is a Patterson-Sullivan probability of dimension δ for c (resp. $\overline{c}$) then the measure

$$\nu_{(\overline{c},c)} := e^{-\delta[\cdot,\cdot]_{(\overline{c},c)}} \mu_{\overline{c}} \otimes \mu_c$$

on $\partial^2_\infty \Gamma$ is Γ-invariant.

Livšic cohomology

Two Hölder cocycles c and c' are said to be *cohomologous* (in the sense of Livšic) if there exists a Hölder continuous function $V : \partial_\infty \Gamma \to \mathbb{R}$ such that for every γ in Γ and x in $\partial_\infty \Gamma$ one has

$$c(\gamma, x) - c'(\gamma, x) = V(\gamma \cdot x) - V(x).$$

In that case the periods of c and c' coincide. Moreover, thanks to the work Livšic [40], cocycles having the same periods are always cohomologous.

Remark 2.3.1. If c and c' are cohomologous cocycles, then there exists a Hölder homeomorphism

$$\mathrm{U}_c\Gamma \to \mathrm{U}_{c'}\Gamma$$

that conjugates the respective translation flows (see [60, Section 3]). Further, if there exists a Patterson-Sullivan probability of dimension δ for c, it is not hard to construct a Patterson-Sullivan probability of dimension δ for c' (similar statement for Gromov products of pairs of dual cohomologous cocycles).

◇

2.3.2 Hopf coordinates for $\mathrm{U}_\rho\Gamma$ and distribution of periodic orbits

Projective Busemann cocycles for ρ

An important family of Hölder cocycles associated to ρ, used by Sambarino in [60, 61] to study asymptotic properties of the Cartan projection of $\rho(\Gamma)$, is the following. Let τ be a point in X_G and $\|\cdot\|_\tau$ be a norm on V invariant under the action of (the lifted to $\mathrm{SL}(V)$ of) the group K^τ. Set

$$c_\tau^1 : \Gamma \times \partial_\infty\Gamma \to \mathbb{R} : \quad c_\tau^1(\gamma, x) := \log \frac{\|\rho\gamma \cdot v_x\|_\tau}{\|v_x\|_\tau},$$

where v_x is any non zero vector in the line $\xi(x)$. Then c_τ^1 is a Hölder cocycle over Γ called the *projective τ-Busemann cocycle* of ρ. Note that the period of $\gamma \in \Gamma_\mathrm{H}$ is given by

$$p_{c_\tau^1}(\gamma) = \lambda_1(\rho\gamma) > 0$$

and in particular the entropy of c_τ^1 coincides with the entropy h_ρ^1 of ρ. It is therefore positive and finite (Fact 2.2.4).

Remark 2.3.2. Let $\mathfrak{a}^+$ be a closed Weyl chamber of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$ for some Cartan subspace $\mathfrak{a} \subset \mathfrak{p}^\tau$. Recall that β^τ denotes the τ-Busemann cocycle of G (Subsection 1.6.1) and that $\varepsilon_1 \in \mathfrak{a}^*$ is the functional defined in Example 1.4.2. One has the equality

$$c_\tau^1(\gamma, x) = \varepsilon_1(\beta^\tau(\rho\gamma, \xi(x)))$$

for every $\gamma \in \Gamma$ and $x \in \partial_\infty\Gamma$.

$\diamond$

Denote $\mathrm{U}_\tau^1\Gamma := \mathrm{U}_{c_\tau^1}\Gamma$.

Lemma 2.3.3 (Hopf parametrization)**.** *The action of Γ on $\tilde{\mathrm{U}}\Gamma$ induced by the Hölder cocycle c_τ^1 is properly discontinuous and co-compact. Further, there exists a Hölder homeomorphism*

$$\mathrm{U}_\rho\Gamma \to \mathrm{U}_\tau^1\Gamma$$

that conjugates the flows ϕ_t and ψ_t. Finally, for every point $(x_0, y_0, t_0) \in \mathrm{U}_\tau^1\Gamma$ the central unstable and strong stable leaves of ψ_t through (x_0, y_0, t_0) are given respectively by

$$\mathscr{W}^{cu}(x_0, y_0, t_0) = \{(x_0, y, t) \in \mathrm{U}_\tau^1\Gamma : \quad y \in \partial_\infty\Gamma \setminus (\Gamma \cdot x_0) \text{ and } t \in \mathbb{R}\}$$

and

$$\mathscr{W}^{ss}(x_0, y_0, t_0) = \{(x, y_0, t_0) \in \mathrm{U}^1_\tau\Gamma : \quad x \in \partial_\infty\Gamma \setminus (\Gamma \cdot y_0)\}.$$

Proof. The map $F_\rho \to \tilde{\mathrm{U}\Gamma}$ defined by

$$(x, y, \vartheta, v) \mapsto (x, y, -\log \|v\|_\tau)$$

is Hölder continuous, injective and equivariant. Moreover one can prove that it is proper and surjective, hence a homeomorphism. The statement involving the flows and the laminations is straightforward.

□

Patterson-Sullivan measures for c^1_τ

Fix a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ for a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}^\tau$ and let

$$\delta^1_\rho := \limsup_{t\to\infty} \frac{\log \# \{\gamma \in \Gamma : \quad a^\tau_1(\rho\gamma) \leq t\}}{t} \geq 0$$

be the *projective critical exponent* of ρ. We now briefly discuss a construction of a Patterson-Sullivan probability μ^1_τ of dimension δ^1_ρ for the cocycle c^1_τ.

There are several ways of showing the existence of μ^1_τ. In the case in which $\rho(\Gamma)$ is Zariski dense one can apply directly the work of Quint [56]. Another approach could be to apply the thermodynamical formalism to the flow $\mathrm{U}^1_\tau\Gamma$ (this is explained in C. [14, Appendix A]). However, because of the contraction properties of ρ, Patterson's method [49] works correctly in our seting[8]. Indeed, one can suppose that the *Poincaré series*

$$\Phi^1_\tau(s) := \sum_{\gamma\in\Gamma} e^{-s a^\tau_1(\rho\gamma)}$$

diverges at $s = \delta^1_\rho$. For every $s > \delta^1_\rho$ consider the measure[9]

$$\frac{1}{\Phi^1_\tau(s)} \sum_{\gamma\in\Gamma} e^{-s a^\tau_1(\rho\gamma)} \delta_{U^\tau_1(\rho\gamma)}.$$

Any weak-star limit as $s \to \delta^1_\rho$ of this familiy of measures (pushed back to $\partial_\infty\Gamma$ by the limit map ξ) satisfies the desired properties. For details, see e.g. Pozzetti-Sambarino-Wienhard [54, Lemma 5.11] and Quint [56, Lemme 6.6].

Let μ^1_τ be then a Patterson-Sullivan probability of dimension δ^1_ρ for the cocycle c^1_τ. The following property of μ^1_τ will be important in the future (notably in the proof of Proposition 5.1.8). The proof presented here is adapted from Glorieux-Monclair [23, Proposition 4.3].

[8]This is the method used by Ledrappier [38] for his general construction.
[9]If x is a point in a metric space X, the symbol δ_x denotes the Dirac mass at x.

Lemma 2.3.4. *The measure μ^1_τ has no atoms.*

Proof. Recall that $\delta^1_\rho \geq 0$, so in fact it is strictly positive. Suppose by contradiction that there exists an atom $y \in \partial_\infty\Gamma$ for μ^1_τ. Then the point y cannot be fixed by any element of Γ and therefore

$$1 = \mu^1_\tau(\partial_\infty\Gamma) \geq \sum_{\gamma\in\Gamma} e^{-\delta^1_\rho c^1_\tau(\gamma^{-1},y)}\mu^1_\tau(y). \tag{2.3.4}$$

Claim 2.3.5. There exists a sequence $\gamma_n \to \infty$ such that $c^1_\tau(\gamma_n^{-1}, y) \to -\infty$.

Proof of Claim 2.3.5. Let x be a point in $\partial_\infty\Gamma$ different from y and take a sequence $\gamma_n \to \infty$ such that

$$(\gamma_n)_+ \to y \text{ and } (\gamma_n)_- \to x.$$

By taking a subsequence if necessary we may assume that γ_n converges uniformly to y on compact sets of $\partial_\infty\Gamma \setminus \{x\}$ (c.f. Bowditch [8, Lemma 2.11]). Let $B(x) \subset \partial_\infty\Gamma$ be the complement of a small neighbourhood of x in $\partial_\infty\Gamma$ and $b(y) \subset B(x)$ be a small neighbourhood of y. Then we can suppose that the inclusion $\gamma_n \cdot B(x) \subset b(y)$ holds for every n.

By Proposition 2.1.3 there exists $\varepsilon > 0$ such that for all n one has

$$\xi(B(x)) \subset B_\varepsilon(S^\tau_{d-1}(\rho\gamma_n)),$$

where $S^\tau_{d-1}(\rho\gamma_n)$ is the Cartan repellor of $\rho\gamma_n$ in $\mathsf{Gr}_{d-1}(V)$. Take a positive A with the following property: for every n and every vector v in the set $B_\varepsilon(S^\tau_{d-1}(\rho\gamma_n))$ one has

$$\|\rho\gamma_n \cdot v\|_\tau \geq A\|\rho\gamma_n\|_\tau\|v\|_\tau.$$

Let $v \neq 0$ be a vector in $\xi(y)$. We have that $\rho\gamma_n^{-1} \cdot v$ belongs to $B_\varepsilon(S^\tau_{d-1}(\rho\gamma_n))$ hence

$$\rho\gamma_n^{-1} \cdot v \to 0$$

as $n \to \infty$. The divergence $c^1_\tau(\gamma_n^{-1}, y) \to -\infty$ follows. □

A combination of equation (2.3.4) and Claim 2.3.5 yields the desired contradiction. □

The Bowen-Margulis probability on $\mathrm{U}^1_\tau\Gamma$

Note that the dual representation

$$\overline{\rho} : \Gamma \to \mathrm{PSL}(V^*)$$

is also projective Anosov and hence the previous construction applies. The projective τ-Busemann cocycle for $\overline{\rho}$ is dual to c^1_τ and it is explicitely given by

$$\overline{c}^1_\tau(\gamma, x) := \log \frac{\|\rho\gamma \cdot \vartheta_x\|^*_\tau}{\|\vartheta_x\|^*_\tau},$$

for any non zero linear functional ϑ_x in $\eta(x)$ (here $\|\cdot\|^*_\tau$ denotes the operator norm on V^* associated to the norm $\|\cdot\|_\tau$ on V). Note that one has the equality

$$\delta^1_\rho = \limsup_{t\to\infty} \frac{\log \#\{\gamma \in \Gamma : \quad a^\tau_1(\overline{\rho}\gamma) \leq t\}}{t}$$

and therefore we have a Patterson-Sullivan probability $\overline{\mu}^1_\tau := \mu_{\overline{c}^1_\tau}$ of dimension δ^1_ρ for $\overline{c}^1_\tau$.

Recall the definition of the projective τ-Gromov product $\mathbb{G}^1_\tau$ from Subsection B.1.1. The map

$$[x, y]^1_\tau := \mathbb{G}^1_\tau(\eta(x), \xi(y))$$

is a Gromov product for the pair $(\overline{c}^1_\tau, c^1_\tau)$.

We now explain why Sambarino's results [60] still hold in our present setting[10].

Proposition 2.3.6 (Sambarino [60, Theorem 3.2] and [60, Theorem 6.5])**.** *The number δ^1_ρ coincides with the topological entropy h^1_ρ of ρ and the measure*

$$e^{-h^1_\rho[\cdot,\cdot]^1_\tau}\overline{\mu}^1_\tau \otimes \mu^1_\tau \otimes dt$$

induces a measure on the quotient space $\mathrm{U}^1_\tau\Gamma$ proportional to the Bowen-Margulis probability of ψ_t.

Proof. From explicit computations one can show that

$$e^{-\delta^1_\rho[\cdot,\cdot]^1_\tau}\overline{\mu}^1_\tau \otimes \mu^1_\tau \otimes dt$$

coincides with a product of measures ν^{cu}_{loc} and ν^{ss}_{loc} as in Fact A.2.1 of Appendix A. □

[10] Recall that [60] deals with the case in which Γ is the fundamental group of a closed negatively curved manifold, while here we only assume that Γ is a word hyperbolic group admitting an Anosov representation.

For a metric space X we denote by $C_c^*(X)$ the dual of the space of compactly supported real continuous functions on X equipped with the weak-star topology. The following is a consequence of Fact 2.2.4 and Proposition 2.3.6.

Proposition 2.3.7 (Sambarino [60, Proposition 4.3])**.** *There exists a positive constant* $\mathtt{m} = \mathtt{m}_{\rho,\tau}$ *such that*

$$\mathtt{m} e^{-h_\rho^1 t} \sum_{\gamma \in \Gamma_{\mathrm{H}}, \lambda_1(\rho\gamma) \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \to e^{-h_\rho^1 [\cdot,\cdot]_\tau^1} \overline{\mu}_\tau^1 \otimes \mu_\tau^1$$

as $t \to \infty$ *on* $C_c^*(\partial_\infty^2 \Gamma)$*.*

□

As mentioned earlier, once Proposition 2.3.7 is established the method of Roblin [58] adapts to show that the function

$$t \mapsto \#\{\gamma \in \Gamma : \quad \log \|\rho\gamma\|_\tau \leq t\}$$

is equivalent to a purely exponential function as $t \to \infty$ (see [60, Theorem 6.5]). This is the type of techniques that we use in this thesis to obtain our counting theorems.

Remark 2.3.8. Let c (resp. $\overline{c}$) be a Hölder cocycle cohomologous to c_τ^1 (resp. $\overline{c}_\tau^1$). Then a result analogous to Proposition 2.3.7 holds for the pair $(\overline{c}, c)$ (c.f. Remark 2.3.1). Note however that the constant $\mathtt{m}$ in the statement may change.

◇

2.3.3 The Borel-Anosov case

The results of the previous subsection can be generalized when ρ is further Borel-Anosov. Indeed, assume that this is the case and let

$$\xi : \partial_\infty \Gamma \to \mathsf{F}(V)$$

be its limit map. Denote by $\mathscr{L}_\rho$ the *asymptotic cone* of $\rho(\Gamma)$, introduced by Benoist in [4]. By definition, it is the subset of $\mathfrak{a}^+$ consisting on all possible limits of sequences of the form

$$\frac{a^\tau(\rho\gamma_n)}{t_n} \tag{2.3.5}$$

where $t_n \to \infty$. Benoist [4, 5] showed that, for Zariski dense subgroups of G, the set $\mathscr{L}_\rho$ coincides with the smallest closed cone containing $\lambda(\rho(\Gamma))$ (which is also showed to be convex and with non empty interior).

Let

$$\mathscr{L}_\rho^* := \{\varphi \in \mathfrak{a}^* : \quad \varphi|_{\mathscr{L}_\rho} \geq 0\}$$

be the *dual cone* and fix a linear functional $\varphi \in \mathscr{L}_\rho^*$. By equation (1.4.1) one has $\varphi(\lambda(\rho\gamma)) \geq 0$ for every $\gamma \in \Gamma$. The φ-*entropy* of ρ is defined by

$$h_\rho^\varphi := \lim_{t\to\infty} \frac{\log \# \{[\gamma] \in [\Gamma] : \quad \gamma \text{ is primitive and } \varphi(\lambda(\rho\gamma)) \leq t\}}{t}.$$

Suppose further that φ belongs to the interior of $\mathscr{L}_\rho^*$, that is, the restriction $\varphi|_{\mathscr{L}_\rho}$ is strictly positive. In this case one can find positive constants $d < D$ for which the inequalities

$$d\varepsilon_1(X) \leq \varphi(X) \leq D\varepsilon_1(X) \tag{2.3.6}$$

hold for every $X \in \mathscr{L}_\rho$. Since the entropy $h_\rho^1 = h_\rho^{\varepsilon_1}$ of ρ is positive and finite, we conclude that the φ-entropy of ρ is also positive and finite.

On the other hand, equation (2.3.6) implies

$$\varphi(\lambda(\rho\gamma)) \geq d\lambda_1(\rho\gamma)$$

for every $\gamma \in \Gamma_{\mathrm{H}}$. Therefore there exists a strictly positive Hölder continuous function

$$f := f^\varphi : \mathrm{U}_\rho\Gamma \to \mathbb{R}$$

for which

$$\int_{[\gamma]} f^\varphi = \varphi(\lambda(\rho\gamma))$$

holds for every $\gamma \in \Gamma_{\mathrm{H}}$ (see Potrie-Sambarino [52, Corollary 4.5] and Sambarino [60, Lemma 3.8]). Let ϕ_t^f be the Hölder reparametrization of the geodesic flow of ρ by the function f (see Section A.3 for a quick reminder on this notion). By Proposition A.3.1 and Fact A.1.1 the topological entropy of ϕ_t^f coincides with h_ρ^φ.

We finish this part with a description of the Bowen-Margulis probability of ϕ_t^f by means of Patterson-Sullivan theory as in [60, Theorem 3.2] and a distribution statement as in [60, Proposition 4.3]. In order to do that, note that there exists a one to one correspondence between the set of Γ-invariant finite Radon measures on $\partial_\infty^2\Gamma$ and the set of ϕ_t^f-invariant finite Radon measures on $\mathrm{U}_\rho\Gamma$. If ν is a Γ-invariant measure on $\partial_\infty^2\Gamma$, the corresponding ϕ_t^f-invariant measure on $\mathrm{U}_\rho\Gamma$ is denoted by m_ν.

The (φ, τ)-*Busemann cocycle* of ρ is defined by

$$c_\tau^\varphi : \Gamma \times \partial_\infty\Gamma \to \mathbb{R} : \quad c_\tau^\varphi(\gamma, x) := \varphi(\beta^\tau(\rho\gamma, \xi(x))),$$

and we consider the dual cocyle

$$\overline{c}_\tau^\varphi : \Gamma \times \partial_\infty \Gamma \to \mathbb{R} \quad \overline{c}_\tau^\varphi(\gamma, x) := \varphi(\iota_{\mathfrak{a}^+} \circ \beta^\tau(\rho\gamma, \xi(x))).$$

Note that for every $\gamma \in \Gamma_{\mathrm{H}}$ one has $p_{c_\tau^\varphi}(\gamma) = \varphi(\lambda(\rho\gamma))$.

The map

$$[\cdot,\cdot]_\tau^\varphi : \partial_\infty^2 \Gamma \to \mathbb{R} : \quad [x, y]_\tau^\varphi := \varphi(\mathbb{G}_\tau(\xi(x), \xi(y))),$$

where $\mathbb{G}_\tau$ is the τ-Gromov product of Subsection B.2.1, is a Gromov product for the pair $(\overline{c}_\tau^\varphi, c_\tau^\varphi)$. Let

$$\delta_\rho^\varphi := \limsup_{t\to\infty} \frac{\log \# \{\gamma \in \Gamma : \quad \varphi(a^\tau(\rho\gamma)) \leq t\}}{t}$$

be the *φ-critical exponent* of ρ. As in the previous subsection, we have a Patterson-Sullivan probality μ_τ^φ (resp. $\overline{\mu}_\tau^\varphi$) for c_τ^φ (resp. $\overline{c}_\tau^\varphi$) of dimension δ_ρ^φ.

With the same proof of Proposition 2.3.6 (see also Remark A.2.2) one has the following.

Proposition 2.3.9 (Sambarino [60, Theorem 3.2] and [60, Theorem 7.13])**.** *The number δ_ρ^φ coincides with the topological entropy h_ρ^φ of ϕ_t^f and the measure m_ν, for*

$$\nu := e^{-h_\rho^\varphi [\cdot,\cdot]_\tau^\varphi} \overline{\mu}_\tau^\varphi \otimes \mu_\tau^\varphi,$$

is proportional to the Bowen-Margulis probability of ϕ_t^f.

□

As in Proposition 2.3.7 one obtains the following.

Proposition 2.3.10 (Sambarino [60, Proposition 4.3])**.** *Assume that ρ is Zariski dense. There exists a positive constant $\tilde{\mathtt{m}} = \tilde{\mathtt{m}}_{\rho,\tau,\varphi}$ such that*

$$\tilde{\mathtt{m}} e^{-h_\rho^\varphi t} \sum_{\gamma \in \Gamma_{\mathrm{H}}, \varphi(\lambda(\rho\gamma)) \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \to e^{-h_\rho^\varphi [\cdot,\cdot]_\tau^\varphi} \overline{\mu}_\tau^\varphi \otimes \mu_\tau^\varphi$$

as $t \to \infty$ on $C_c^*(\partial_\infty^2 \Gamma)$.

□

Remark 2.3.11. Let c (resp. $\overline{c}$) be a Hölder cocycle cohomologous to c_τ^φ (resp. $\overline{c}_\tau^\varphi$). Then a result analogous to Proposition 2.3.10 holds for the pair $(\overline{c}, c)$.

◇

Part II

Counting for $\mathrm{G} = \mathrm{PSO}(p, q)$ and $\mathrm{H} = \mathrm{PSO}(p, q-1)$

What we present here is the subject of the article C. [14].

Throughout this part of the thesis, G denotes the Lie group $\mathrm{PSO}(p,q)$ for some integers $p \geq 1$ and $q \geq 2$. The goal here is two folded. On the one hand, let o be a point in the pseudo-Riemannian hyperbolic space $\mathsf{H}^{p,q-1}$. We introduce the subset

$$\mathscr{C}^{>}_{o,\mathrm{G}}$$

consisting on elements of G that map the basepoint into a point that can be joined to o by a *space-like geodesic*, i.e. a geodesic tangent to positive vectors for the Killing metric on $\mathsf{H}^{p,q-1}$. We show that a decomposition of $\mathscr{C}^{>}_{o,\mathrm{G}}$ analogue to the classical Cartan decomposition of G holds: one has the equality

$$\mathscr{C}^{>}_{o,\mathrm{G}} = \mathrm{H}^o \exp(\mathfrak{b}^+)\mathrm{H}^o$$

and this allow us to define a projection

$$b^o : \mathscr{C}^{>}_{o,\mathrm{G}} \to \mathfrak{b}^+$$

as in the Riemannian setting (this is done in Chapter 3). In Proposition 3.2.2 we provide geometric interpretations for the norm of $b^o(g)$ for a given element g in $\mathscr{C}^{>}_{o,\mathrm{G}}$: on the one side, it coincides with the distance between S^o and its image by g and, on the other, it is the length of the (space-like) geodesic segment in $\mathsf{H}^{p,q-1}$ joining o with $g \cdot o$. Further, in Proposition 3.2.4 we give a concrete linear algebraic method to calculate the norm of $b^o(g)$: it is (one half of) the spectral radius of $\sigma^o(g)g^{-1}$.

The second goal of this part is to find a simple asymptotic of the function

$$t \mapsto \#\{\gamma \in \Gamma : \quad \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t\} \tag{2.3.7}$$

as $t \to \infty$. Here $\rho : \Gamma \to \mathrm{G}$ is a $\mathrm{P}_1^{p,q}$-Anosov representation and o is a basepoint in the set

$$\boldsymbol{\Omega}_\rho := \{o \in \mathsf{H}^{p,q-1} : \quad \langle o, \xi(x)\rangle_{p,q} \neq 0 \text{ for all } x \in \partial_\infty\Gamma\},$$

where ξ is the limit map of ρ in projective space. The key results for obtaining this counting result are Lemma 4.3.3 (item *4.*) and Lemma 5.1.6. Indeed, Proposition 3.2.4 tell us that we need to estimate the spectral radius of the product of the $((r,\varepsilon)$-proximal) elements $\sigma^o(\rho\gamma)$ and $\rho\gamma^{-1}$. The fact that o belongs to $\boldsymbol{\Omega}_\rho$ will give us a "genericity" condition between attractors and repellors of these elements. Benoist's Theorem B.1.5 provides a precise estimate for the spectral radius of the product $\sigma^o(\rho\gamma)\rho\gamma^{-1}$ under these conditions. We then find a Hölder cocycle c_o over $\partial_\infty\Gamma$, analogue to the projective τ-Busemann cocycle of ρ, for which the associated Gromov product coincides with the projective cross-ratio of attractors and repellors of $\sigma^o(\rho\gamma)$ and $\rho\gamma^{-1}$. Equipped with these definitions, we are in position to apply Sambarino's adaptation [60] of Roblin's method [58] in our concrete framework to obtain the desired result (Proposition 5.1.10).

Finally, in Subsection 5.2.3 we apply similar ideas to show an asymptotic for a counting function similar to the given by equation (2.3.7), but with b^o replaced by the polar projection $b^{o,\tau}$ introduced in Section 1.4.

Chapter 3

Generalized Cartan decomposition

3.1 Geometric preliminaries

We begin with some well known geometric facts about the space $\mathsf{H}^{p,q-1}$. Even though this section is intended mostly to fix some terminology and notations, Subsection 3.1.3 will be particularly important in the future: for a basepoint $o \in \mathsf{H}^{p,q-1}$, we describe the unique open orbit of H^o acting on the space of isotropic lines $\partial\mathsf{H}^{p,q-1}$ in several equivalent ways. These equivalences will be used repeatedly in the sequel.

3.1.1 Geodesics of $\mathsf{H}^{p,q-1}$

One can see that geodesics of $\mathsf{H}^{p,q-1}$ coincide with the intersections of straight lines of $\mathsf{P}(\mathbb{R}^{p,q})$ with $\mathsf{H}^{p,q-1}$. They are divided in three types:

- *Space-like geodesics:* associated to 2-dimensional subspaces of $\mathbb{R}^{p,q}$ on which $\langle\cdot,\cdot\rangle_{p,q}$ has signature $(1,1)$. They have positive speed and meet the boundary $\partial\mathsf{H}^{p,q-1}$ in two distinct points.

- *Time-like geodesics:* associated to 2-dimensional subspaces of $\mathbb{R}^{p,q}$ on which $\langle\cdot,\cdot\rangle_{p,q}$ has signature $(0,2)$. They have negative speed and do not meet the boundary (they are closed).

- *Light-like geodesics:* associated to 2-dimensional subspaces of $\mathbb{R}^{p,q}$ on which $\langle\cdot,\cdot\rangle_{p,q}$ has signature $(0,1)$, that is, it is degenerate but has a negative eigenvalue. They have zero speed and meet the boundary in a single point.

For a point $o \in \mathsf{H}^{p,q-1}$ we denote by $\mathscr{C}_o^0$ (resp. $\mathscr{C}_o^>$) the set of points of $\mathsf{H}^{p,q-1}$ that can be joined with o by a light-like (resp. space-like) geodesic. Its closure in $\mathsf{P}(\mathbb{R}^{p,q})$ is denoted by $\overline{\mathscr{C}_o^0}$ (resp. $\overline{\mathscr{C}_o^>}$). For a point o' in $\mathscr{C}_o^>$

we denote by $\mathtt{l}_{o,o'}$ the length of the geodesic segment connecting the point o with o'.

3.1.2 Light-cones

The following lemma is proved by Glorieux-Monclair in [23, Lemma 2.2].

Lemma 3.1.1. *Let $o \in \mathsf{H}^{p,q-1}$. Then $\overline{\mathscr{C}_o^0} \cap \partial\mathsf{H}^{p,q-1} = o^{\perp_{p,q}} \cap \partial\mathsf{H}^{p,q-1}$.*

□

3.1.3 End points of space-like geodesics

Let o be a point in $\mathsf{H}^{p,q-1}$ and recall that J^o denotes the matrix

$$J^o = \mathrm{id}_o \oplus (-\mathrm{id}_{o^{\perp_{p,q}}}).$$

Note that J^o preserves the form $\langle\cdot,\cdot\rangle_{p,q}$ and therefore acts on $\partial\mathsf{H}^{p,q-1}$. Let

$$(\partial\mathsf{H}^{p,q-1})^o := \{\xi \in \partial\mathsf{H}^{p,q-1} : \quad J^o \cdot \xi \neq \xi\}.$$

This set is the unique open orbit of the action $\mathrm{H}^o \curvearrowright \partial\mathsf{H}^{p,q-1}$ and it is not the whole boundary of $\mathsf{H}^{p,q-1}$.

Proposition 3.1.2. *Let $o \in \mathsf{H}^{p,q-1}$. Then the following equalities hold:*

$$\begin{aligned}(\partial\mathsf{H}^{p,q-1})^o &= \{\xi \in \partial\mathsf{H}^{p,q-1} : \quad J^o \cdot \xi \notin \xi^{\perp_{p,q}}\}\\ &= \partial\mathsf{H}^{p,q-1} \setminus o^{\perp_{p,q}}\\ &= \partial\mathsf{H}^{p,q-1} \setminus \overline{\mathscr{C}_o^0}.\end{aligned}$$

Proof. The equality $\partial\mathsf{H}^{p,q-1}\setminus o^{\perp_{p,q}} = \partial\mathsf{H}^{p,q-1}\setminus\overline{\mathscr{C}_o^0}$ is a consequence of Lemma 3.1.1. The other equalities follow from definitions.

□

3.2 Cartan decomposition

Define

$$\mathscr{C}_{o,\mathrm{G}}^{>} := \{g \in \mathrm{G} : \quad g \cdot o \in \mathscr{C}_o^{>}\}.$$

Fix an element $\tau \in \mathsf{S}^o$, a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ and a closed Weyl chamber $\mathfrak{b}^+ \subset \mathfrak{b}$. Recall that $\mathfrak{b}$ is in this case one dimensional and that $\mathfrak{b}^+$ is a ray in $\mathfrak{b}$.

Proposition 3.2.1. *For every g in $\mathscr{C}_{o,\mathrm{G}}^{>}$ one can write*

$$g = h \exp(X) h'$$

for some $h, h' \in \mathrm{H}^o$ *and a unique* $X \in \mathfrak{b}^+$.

Note that since H^o acts on $T_o\mathsf{H}^{p,q-1}$ and preserves the pseudo-Riemannian structure, this decomposition of g can only hold when $g \in \mathscr{C}^{>}_{o,\mathrm{G}}$.

Proof of Proposition 3.2.1. Take h in H^o such that $h^{-1}g \cdot o \in \exp(\mathfrak{b}^+) \cdot o$ (c.f. Remark 1.2.3). There exists then $X \in \mathfrak{b}^+$ and $h' \in \mathrm{H}^o$ such that $h^{-1}g = \exp(X)h'$. Note that X is unique since it is determined by the length of the geodesic segment connecting o with $g \cdot o$. □

We define the map

$$b^o : \mathscr{C}^{>}_{o,\mathrm{G}} \to \mathfrak{b}^+ : \quad g \mapsto b^o(g) \tag{3.2.1}$$

where $g = h \exp(b^o(g))h'$ for some $h, h' \in \mathrm{H}^o$. Note that b^o descends to the quotient $\mathscr{C}^{>}_o \subset \mathsf{H}^{p,q-1}$ but this map is not proper (compare with Remark 1.4.3).

Proposition 3.2.2. *For every* g *in* $\mathscr{C}^{>}_{o,\mathrm{G}}$ *one has*

$$\mathtt{l}_{o,g\cdot o} = \|b^o(g)\|_{\mathfrak{b}} = d_{\mathsf{X}_\mathrm{G}}(\mathsf{S}^o, g \cdot \mathsf{S}^o).$$

Proof. The first equality was already discussed in the proof of Proposition 3.2.1. For the second one write $g = h \exp(b^o(g))h'$. Since S^o is H^o-invariant we have

$$d_{\mathsf{X}_\mathrm{G}}(\mathsf{S}^o, h \exp(b^o(g))h' \cdot \mathsf{S}^o) = d_{\mathsf{X}_\mathrm{G}}(\mathsf{S}^o, \exp(b^o(g)) \cdot \mathsf{S}^o).$$

Set $X := b^o(g)$. If $X = 0$ there is nothing to prove, so assume $X \neq 0$. In that case S^o is disjoint from $\exp(X) \cdot \mathsf{S}^o$: since the action of $\mathfrak{b}$ on the geodesic $\exp(\mathfrak{b}) \cdot \tau$ is free, this follows from the fact that X_G is non positively curved and the fact that $\exp(\mathfrak{b}) \cdot \tau$ intersects orthogonally S^o (resp. $\exp(X) \cdot \mathsf{S}^o$) in τ (resp. $\exp(X) \cdot \tau$).

Claim 3.2.3. Take $\tau' \in \mathsf{S}^o$ and $\tau'' \in \exp(X) \cdot \mathsf{S}^o$. Then the following holds:

$$d_{\mathsf{X}_\mathrm{G}}(\tau', \tau'') \geq d_{\mathsf{X}_\mathrm{G}}(\tau, \exp(X) \cdot \tau).$$

Proof of Claim 3.2.3. Let $\beta_1 \subset \mathsf{S}^o$ (resp. $\beta_2 \subset \exp(X) \cdot \mathsf{S}^o$) be the unit speed geodesic connecting $\beta_1(0) = \tau$ (resp. $\beta_2(0) = \exp(X) \cdot \tau$) with τ' (resp. τ''). Then β_1 and β_2 are disjoint and from the fact that X_G is non positively curved follows that the map

$$(t, s) \mapsto d_{\mathsf{X}_\mathrm{G}}(\beta_1(t), \beta_2(s))$$

is smooth (see Petersen [50, p.129]). Moreover, since $\exp(\mathfrak{b})\cdot\tau$ is orthogonal both to S^o and $\exp(X)\cdot\mathsf{S}^o$ we conclude that the differential at $(0,0)$ of this map is zero.

Take $t_0>0$ such that $\beta_1(t_0)=\tau'$ and a positive a such that the geodesic $t\mapsto\beta_2(at)$ equals τ'' in t_0. By Busemann [13, Theorem 3.6] the map

$$t\mapsto d_{\mathsf{X}_{\mathrm{G}}}(\beta_1(t),\beta_2(at))$$

is convex. Since it has a critical point at $t=0$ the proof of the claim is finished.

□

Since

$$d_{\mathsf{X}_{\mathrm{G}}}(\tau,\exp(X)\cdot\tau)=\|X\|_{\mathfrak{b}}$$

the proof of the proposition follows.

□

Recall that $\lambda_1(g)$ denotes the logarithm of the spectral radius of $g\in\mathrm{G}$.

Proposition 3.2.4. *For every g in $\mathscr{C}^{>}_{o,\mathrm{G}}$ one has*

$$\|b^o(g)\|_{\mathfrak{b}}=\tfrac{1}{2}\lambda_1(J^ogJ^og^{-1}).$$

Proof. Recall from Example 1.2.4 that $\exp(\mathfrak{b})$ is a one parameter subgroup of G consisting on diagonalizable elements. Further, for $X\neq0$ in $\mathfrak{b}$ all the eigenvalues of X are zero, except for two of them which are opposite. If $s>0$ is the unique strictly positive eigenvalue of X then one has[1]

$$\lambda_1(X)=s=\|X\|_{\mathfrak{b}}.$$

Now let $g=h\exp(X)h'$ be a decomposition of g as in Proposition 3.2.1. Since J^o commutes with elements of H^o, the number $\frac{1}{2}\lambda_1(J^ogJ^og^{-1})$ coincides with

$$\tfrac{1}{2}\lambda_1(J^oh\exp(X)J^o\exp(X)^{-1}h^{-1})=\tfrac{1}{2}\lambda_1(J^o\exp(X)J^o\exp(X)^{-1}).$$

But X belongs to $\mathfrak{q}^o$ and therefore we have $J^o\exp(X)^{-1}=\exp(X)J^o$. This completes the proof. □

[1] As in Example 1.4.5 here we are abusing of notations because the equality actually holds up to a scalar positive constant only depending on $d=p+q$.

Chapter 4

Counting functions and first estimates

Let $\rho : \Gamma \to \mathrm{G}$ be a $\mathrm{P}_1^{p,q}$-Anosov representation. As in Section 2.2 we will denote by

$$\xi : \partial_\infty \Gamma \to \partial \mathsf{H}^{p,q-1}$$

its limit map in projective space and by $\eta := \xi^{\perp_{p,q}}$ the dual limit map in the Grassmannian $\mathsf{Gr}_{d-1}(\mathbb{R}^{p,q})$. Also, the image of ξ will be denoted by L_ρ^1.

Define

$$\mathbf{\Omega}_\rho := \{o \in \mathsf{H}^{p,q-1} : \quad J^o \cdot \xi(x) \notin \eta(x) \text{ for all } x \in \partial_\infty \Gamma\}.$$

This chapter is structured as follows. In Section 4.1 we discuss some examples of projective Anosov representations ρ for which the set $\mathbf{\Omega}_\rho$ is non empty. In Section 4.2 we show that if o is a point in $\mathbf{\Omega}_\rho$ then the geodesic connecting o with $\rho\gamma \cdot o$ is space-like (apart from possibly finitely many exceptions $\gamma \in \Gamma$). In Section 4.3 we study the matrices $J^o(\rho\gamma)J^o(\rho\gamma^{-1})$ for a point o in $\mathbf{\Omega}_\rho$: we apply to them Benoist's work on proximality. Finiteness of our counting functions is proved in Section 4.4. Finally, in Section 4.5 we prove a proposition that will be needed in the proof of Theorem A.

4.1 Examples

From Proposition 3.1.2 we know that the following alternative description of $\mathbf{\Omega}_\rho$ holds

$$\mathbf{\Omega}_\rho = \{o = [\hat{o}] \in \mathsf{H}^{p,q-1} : \quad \langle \hat{o}, \hat{\xi} \rangle_{p,q} \neq 0 \text{ for all } \xi = [\hat{\xi}] \in \mathrm{L}_\rho^1\}.$$

In the following we discuss non emptyness of $\mathbf{\Omega}_\rho$ for the examples treated in Example 2.1.10.

Example 4.1.1.

- Fix an integer m such that $p \geq m \geq 2$. Consider a representation ρ_0 given by the composition of the holonomy representation of a convex co-compact hyperbolic manifold of dimension m with the representation

$$\Lambda : \mathrm{SO}(m,1) \to \mathrm{G}$$

induced by the choice of a subspace π of $\mathbb{R}^{p,q}$ of signature $(m,1)$. Let π' be a $(q-1)$-dimensional negative definite subspace of $\mathbb{R}^{p,q}$ contained in $\pi^{\perp_{p,q}}$. Then for any point $o \in \mathsf{H}^{p,q-1}$ for which the subspace $o \oplus \pi'$ is negative definite one has $o \in \mathbf{\Omega}_{\rho_0}$.

- Let $\Gamma < \mathrm{G}$ be a $\mathsf{H}^{p,q-1}$-convex co-compact group (as in Danciger-Guéritaud-Kassel [17, 18]) and $\rho : \Gamma \to \mathrm{G}$ be the inclusion representation. Let Ω be a non empty Γ-invariant properly convex open subset of $\mathsf{H}^{p,q-1}$. By [17, Proposition 4.5], Ω is contained in $\mathbf{\Omega}_\rho$.

- By Schottky type constructions one can find examples of $\mathrm{P}_1^{p,q}$-Anosov representations ρ into G whose image is not $\mathsf{H}^{p,q-1}$-convex co-compact but satisfy $\mathbf{\Omega}_\rho \neq \emptyset$ (see [18, Examples 5.2 & 5.3]).

◇

4.2 Dynamics on $\mathbf{\Omega}_\rho$

The following proposition is well-known, we include a proof for completeness.

Proposition 4.2.1. *Let $\rho : \Gamma \to \mathrm{G}$ be a $\mathrm{P}_1^{p,q}$-Anosov representation. Then the action of Γ on $\mathbf{\Omega}_\rho$ is properly discontinuous, that is, for every compact set $C \subset \mathbf{\Omega}_\rho$ one has*

$$\#\{\gamma \in \Gamma : \quad \rho\gamma \cdot C \cap C \neq \emptyset\} < \infty.$$

Moreover, for any point o in $\mathbf{\Omega}_\rho$ the set of accumulation points of $\rho(\Gamma) \cdot o$ in $\mathsf{H}^{p,q-1} \cup \partial\mathsf{H}^{p,q-1}$ coincides with the limit set L_ρ^1.

Proof. Let $C \subset \mathbf{\Omega}_\rho$ be a compact set and fix a point $\tau \in \mathsf{S}^o$. By definition of $\mathbf{\Omega}_\rho$ we can take a positive ε such that

$$C \cap \bigcup_{x \in \partial_\infty \Gamma} b_\varepsilon(\xi(x)) = \emptyset \text{ and } C \subset \bigcap_{x \in \partial_\infty \Gamma} B_\varepsilon(\eta(x)).$$

By Proposition 2.1.3, Remark 1.5.2 and equation (2.1.1) we know that, apart from possibly finitely many exceptions γ in Γ, the following holds:

$$b_{\frac{\varepsilon}{2}}(U_1^\tau(\rho\gamma)) \subset \bigcup_{x\in\partial_\infty\Gamma} b_\varepsilon(\xi(x)),$$

$$\bigcap_{x\in\partial_\infty\Gamma} B_\varepsilon(\eta(x)) \subset B_{\frac{\varepsilon}{2}}(S_{d-1}^\tau(\rho\gamma))$$

and

$$\rho\gamma\cdot B_{\frac{\varepsilon}{2}}(S_{d-1}^\tau(\rho\gamma)) \subset b_{\frac{\varepsilon}{2}}(U_1^\tau(\rho\gamma)).$$

For these γ we have then that $\rho\gamma\cdot C$ is contained in the ε-neighbourhood of L_ρ^1 and thus is disjoint from C.

We have shown that the action of Γ on $\boldsymbol{\Omega}_\rho$ is properly discontinuous and that for any point o in $\boldsymbol{\Omega}_\rho$ the accumulation points of $\rho(\Gamma)\cdot o$ belong to L_ρ^1. Conversely, the Γ-orbit of any point in L_ρ^1 is dense in the limit set and now the proof is complete.

□

Let $o\in\boldsymbol{\Omega}_\rho$ and recall the notations introduced in Subsection 3.1.1. Given an open set $W\subset\partial\mathsf{H}^{p,q-1}$ disjoint from $\overline{\mathscr{C}_o^0}\cap\partial\mathsf{H}^{p,q-1}$ we denote by $\mathscr{C}_o^{>W}$ the subset of $\mathscr{C}_o^>$ consisting of points o' for which the (space-like) geodesic ray connecting o with o' has its end point in W.

The following corollary has been proved by Glorieux-Monclair [23] under the assumption that ρ is $\mathsf{H}^{p,q-1}$-convex co-compact.

Corollary 4.2.2. *Let $\rho:\Gamma\to\mathrm{G}$ be a $\mathrm{P}_1^{p,q}$-Anosov representation, a point $o\in\boldsymbol{\Omega}_\rho$ and $W\subset\partial\mathsf{H}^{p,q-1}$ an open set containing L_ρ^1 with closure disjoint from $\overline{\mathscr{C}_o^0}\cap\partial\mathsf{H}^{p,q-1}$. Then apart from possibly finitely many exceptions γ in Γ one has $\rho\gamma\cdot o\in\mathscr{C}_o^{>W}$. In particular the geodesic joining o with $\rho\gamma\cdot o$ is space-like.*

Proof. Let C be the closure of $\mathsf{H}^{p,q-1}\setminus\mathscr{C}_o^{>W}$ in $\mathsf{H}^{p,q-1}\cup\partial\mathsf{H}^{p,q-1}$. Note that C is compact and by Proposition 4.2.1 does not contain accumulation points of $\rho(\Gamma)\cdot o$, hence $\rho(\Gamma)\cdot o\cap C$ is finite. Since the map $\gamma\mapsto\rho\gamma\cdot o$ is proper the proof is complete.

□

4.3 Proximality of $J^o(\rho\gamma)J^o(\rho\gamma^{-1})$

For the rest of the section we fix a $\mathrm{P}_1^{p,q}$-Anosov representation $\rho:\Gamma\to\mathrm{G}$, a point $o\in\boldsymbol{\Omega}_\rho$ and a norm $\|\cdot\|_\tau$ induced by the choice of a Cartan involution $\tau\in\mathsf{S}^o$. Let $d(\cdot,\cdot)$ be the induced distance in $\mathsf{P}(\mathbb{R}^{p,q})$ (c.f. Appendix B).

The next lemma is a direct consequence of Proposition 2.1.3, equation (2.1.2) and the definition of $\boldsymbol{\Omega}_\rho$.

Lemma 4.3.1. *There exists a positive constant D such that*

$$\#\{\gamma \in \Gamma : \quad d(J^o \cdot U_1^\tau(\rho\gamma), S_{d-1}^\tau(\rho\gamma^{-1})) < D\} < \infty.$$

□

Lemma 4.3.2. *There exist $0 < \varepsilon \leq r$ such that, apart from possibly finitely many exceptions $\gamma \in \Gamma$, the matrix $J^o(\rho\gamma)J^o(\rho\gamma^{-1})$ is (r, ε)-proximal.*

Proof. We apply a "ping-pong" argument together with Lemma B.1.2. By Lemma 4.3.1 we can take a positive constant r and a finite subset $F \subset \Gamma$ such that for every $\gamma \in \Gamma \setminus F$ one has

$$d(J^o \cdot U_1^\tau(\rho\gamma), S_{d-1}^\tau(\rho\gamma^{-1})) \geq 6r. \tag{4.3.1}$$

Take $0 < \varepsilon \leq r$ such that for every $\gamma \in \Gamma \setminus F$ one has

$$b_\varepsilon(J^o \cdot U_1^\tau(\rho\gamma)) \subset B_\varepsilon(S_{d-1}^\tau(\rho\gamma^{-1})).$$

By Remark 1.2.5 the matrix J^o preserves the distance $d(\cdot, \cdot)$ thus

$$J^o \cdot b_\varepsilon(U_1^\tau(\rho\gamma)) \subset B_\varepsilon(S_{d-1}^\tau(\rho\gamma^{-1})).$$

By taking F larger if necessary we have that

$$\rho\gamma^{-1} \cdot B_\varepsilon(S_{d-1}^\tau(\rho\gamma^{-1})) \subset b_\varepsilon(U_1^\tau(\rho\gamma^{-1}))$$

holds for every γ in $\Gamma \setminus F$. It follows that

$$J^o(\rho\gamma^{-1}) \cdot B_\varepsilon(S_{d-1}^\tau(\rho\gamma^{-1})) \subset B_\varepsilon(S_{d-1}^\tau(\rho\gamma))$$

and applying $\rho\gamma$ we obtain

$$(\rho\gamma)J^o(\rho\gamma^{-1}) \cdot B_\varepsilon(S_{d-1}^\tau(\rho\gamma^{-1})) \subset b_\varepsilon(U_1^\tau(\rho\gamma)).$$

Then

$$J^o(\rho\gamma)J^o(\rho\gamma^{-1}) \cdot B_\varepsilon(S_{d-1}^\tau(\rho\gamma^{-1})) \subset b_\varepsilon(J^o \cdot U_1^\tau(\rho\gamma)).$$

By equation (4.3.1) and Lemma B.1.2 the proof is finished.

□

The following is a strengthening of Lemma 4.3.2. It provides a link between the projections b^o and $b^{o,\tau}$ and the spectral radii of proximal elements in $\rho(\Gamma)$. For the remainder of the section we fix a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ and a closed Weyl chamber $\mathfrak{b}^+ \subset \mathfrak{b}$. Recall from Appendix B that $\mathbb{B}^1$ (resp. $\mathbb{G}_\tau^1$) denotes the projective cross-ratio (resp. projective τ-Gromov product).

Lemma 4.3.3. *Fix any $\delta > 0$ and A and B two compact disjoint sets in $\partial_\infty\Gamma$. Then there exist $0 < \varepsilon \leq r$ such that, apart from possibly finitely many exceptions $\gamma \in \Gamma_{\mathrm{H}}$ with $\gamma_- \in A$ and $\gamma_+ \in B$, the following holds:*

1. *The matrices* $J^o(\rho\gamma)J^o$ *and* $\rho\gamma^{-1}$ *are* (r,ε)*-proximal.*
2. $d(J^o\cdot(\rho\gamma)_+,(\rho\gamma^{-1})_-)\geq 6r$ *and* $d((\rho\gamma^{-1})_+,J^o\cdot(\rho\gamma)_-)\geq 6r$*.*
3. $d((J^o(\rho\gamma)J^o)_+,(\rho\gamma^{-1})_-)\geq 6r$ *and* $d((\rho\gamma^{-1})_+,(J^o(\rho\gamma)J^o)_-)\geq 6r$*.*
4. *The matrix* $\rho\gamma$ *belongs to* $\mathscr{C}^{>}_{o,\mathrm{G}}$ *and the number*
$$\|b^o(\rho\gamma)\|_{\mathfrak{b}}-\lambda_1(\rho\gamma)$$
is at distance at most δ *from*
$$\tfrac{1}{2}\mathbb{B}^1(J^o\cdot(\rho\gamma)_-,J^o\cdot(\rho\gamma)_+,(\rho\gamma^{-1})_-,(\rho\gamma^{-1})_+).$$
5. *The number*
$$\|b^{o,\tau}(\rho\gamma)\|_{\mathfrak{b}}-\lambda_1(\rho\gamma)$$
is at distance at most δ *from*
$$\tfrac{1}{2}\mathbb{B}^1(J^o\cdot(\rho\gamma)_-,J^o\cdot(\rho\gamma)_+,(\rho\gamma^{-1})_-,(\rho\gamma^{-1})_+)-\tfrac{1}{2}\mathbb{G}^1_\tau((\rho\gamma^{-1})_-,J^o\cdot(\rho\gamma)_+).$$

Proof. By transversality condition (2.1.2) there exists $r>0$ such that
$$d(\xi(x),\eta(y))\geq 2r \text{ and } d(\xi(y),\eta(x))\geq 2r \tag{4.3.2}$$
holds for all $(x,y)\in A\times B$. Further, since $o\in\mathbf{\Omega}_\rho$ we may assume
$$d(J^o\cdot\xi(x),\eta(x))\geq 6r \tag{4.3.3}$$
for all $x\in\partial_\infty\Gamma$. Given these $r>0$ and $2\delta>0$, we consider $\varepsilon>0$ as in Benoist's Theorem B.1.5.

By Lemma 2.1.4 there exists a finite subset F of Γ_{H} outside of which elements satisfying $d((\rho\gamma)_+,(\rho\gamma)_-)\geq 2r$ are (r,ε)-proximal. Thanks to equation (4.3.2), for all $\gamma\in\Gamma_{\mathrm{H}}\setminus F$ with $\gamma_-\in A$ and $\gamma_+\in B$ one has that $\rho\gamma^{\pm1}$ is (r,ε)-proximal. Moreover, since $J^o=(J^o)^{-1}$ preserves $\|\cdot\|_\tau$ we have that $J^o(\rho\gamma)J^o$ is (r,ε)-proximal with $(J^o(\rho\gamma)J^o)_\pm=J^o\cdot(\rho\gamma)_\pm$. In fact, by equation (4.3.3) we have
$$d(J^o\cdot(\rho\gamma)_+,(\rho\gamma^{-1})_-)\geq 6r \text{ and } d((\rho\gamma^{-1})_+,J^o\cdot(\rho\gamma)_-)\geq 6r.$$
Thanks to Proposition 3.2.4 (and Corollary 4.2.2), Example 1.4.5, Theorem B.1.5 and the fact that $\lambda_1(\rho\gamma^{-1})$ equals $\lambda_1(\rho\gamma)$ for all γ in Γ, the proof is finished.

□

4.4 Orbital counting functions

Proposition 4.4.1. *For every $t \geq 0$ one has*

$$\#\left\{\gamma \in \Gamma : \quad \|b^{o,\tau}(\rho\gamma)\|_{\mathfrak{b}} \leq t\right\} < \infty.$$

Proof. By Remark 1.4.3 the map $b^{o,\tau}$ descends to a proper map in $\mathsf{H}^{p,q-1} \cong \mathrm{G}/\mathrm{H}^o$, that we still denote by $b^{o,\tau}$. Hence

$$C := \{o' \in \mathsf{H}^{p,q-1} : \quad \|b^{o,\tau}(o')\|_{\mathfrak{b}} \leq t\}$$

is compact. By Proposition 4.2.1, apart from possibly finitely many exceptions γ in Γ, we have that $\rho\gamma \cdot o$ does not belong to C.

□

The next proposition follows from a combination of Proposition 3.2.4, Example 1.4.5, Lemmas 4.3.2 and B.1.1, and the previous proposition.

Proposition 4.4.2. *For every $t \geq 0$ one has*

$$\#\left\{\gamma \in \Gamma : \quad \rho\gamma \in \mathscr{C}^{>}_{o,\mathrm{G}} \text{ and } \|b^{o}(\rho\gamma)\|_{\mathfrak{b}} \leq t\right\} < \infty.$$

□

Remark 4.4.3. Assume that ρ is $\mathsf{H}^{p,q-1}$-convex co-compact and that the basepoint o belongs to the convex hull of the limit set of ρ. By Corollary 4.2.2 and Proposition 3.2.2 we have that

$$\limsup_{t\to\infty} \frac{\log \#\{\gamma \in \Gamma : \quad \rho\gamma \in \mathscr{C}^{>}_{o,\mathrm{G}} \text{ and } \|b^{o}(\rho\gamma)\|_{\mathfrak{b}} \leq t\}}{t}$$

coincides with

$$\limsup_{t\to\infty} \frac{\log \#\{\gamma \in \Gamma : \quad d_{\mathsf{H}^{p,q-1}}(o, \rho\gamma \cdot o) \leq t\}}{t},$$

where $d_{\mathsf{H}^{p,q-1}}(\cdot,\cdot)$ is the $\mathsf{H}^{p,q-1}$-distance introduced in [23].

◇

4.5 Weak triangle inequality

The following proposition is inspired by Glorieux-Monclair [23, Theorem 3.5].

Proposition 4.5.1. *There exists a constant $L > 0$ such that for every $\gamma_0 \in \Gamma$ there exists $D_{\gamma_0} > 0$ with the following property: for every $\gamma \in \Gamma$ with $|\gamma|_\Gamma > L$ one has*

$$\tfrac{1}{2}\lambda_1(J^o(\rho\gamma_0)(\rho\gamma)J^o(\rho\gamma^{-1})(\rho\gamma_0^{-1}) \leq D_{\gamma_0} + \tfrac{1}{2}\lambda_1(J^o(\rho\gamma)J^o(\rho\gamma^{-1})).$$

We can think about the content of Proposition 4.5.1 as follows. Fix $\gamma_0 \in \Gamma$ such that $\rho\gamma_0 \in \mathscr{C}^{>}_{o,\mathrm{G}}$. By Corollary 4.2.2, apart from possibly finitely many exceptions γ in Γ one has $\rho\gamma \in \mathscr{C}^{>}_{o,\mathrm{G}}$ and $(\rho\gamma_0)(\rho\gamma) \in \mathscr{C}^{>}_{o,\mathrm{G}}$. Thanks to Propositions 3.2.2 and 3.2.4, the inequality established in Proposition 4.5.1 can be stated as

$$\mathtt{l}_{o,(\rho\gamma_0)(\rho\gamma)\cdot o} \leq D_{\gamma_0} + \mathtt{l}_{\rho\gamma_0\cdot o,(\rho\gamma_0)(\rho\gamma)\cdot o},$$

where the constant D_{γ_0} depends on the choice of o and γ_0 (and ρ) but not on the choice of γ. Even though the function $\mathtt{l}_{\cdot,\cdot}$ is not a distance, we can heuristically think about D_{γ_0} as the term that replaces $\mathtt{l}_{o,\rho\gamma_0\cdot o}$ in the usual triangle inequality for distances.

Proof of Proposition 4.5.1. Take $0 < \varepsilon \leq r$ as in Lemma 4.3.2. Let $L > 0$ such that for every γ in Γ with $|\gamma|_\Gamma > L$ the matrix $J^o(\rho\gamma)J^o(\rho\gamma^{-1})$ is (r, ε)-proximal. Fix $\gamma_0 \in \Gamma$ and let γ be a element in Γ with $|\gamma|_\Gamma > L$. We have

$$\tfrac{1}{2}\lambda_1(J^o(\rho\gamma_0)(\rho\gamma)J^o(\rho\gamma^{-1})(\rho\gamma_0^{-1})) \leq \tfrac{1}{2}\log\|J^o(\rho\gamma_0)(\rho\gamma)J^o(\rho\gamma^{-1})(\rho\gamma_0^{-1})\|_\tau.$$

By Remark 1.2.5 the right side number equals

$$\tfrac{1}{2}\log\|(\rho\gamma_0)(\rho\gamma)J^o(\rho\gamma^{-1})(\rho\gamma_0^{-1})\|_\tau$$

which is less than or equal to

$$D'_{\gamma_0} + \tfrac{1}{2}\log\|J^o(\rho\gamma)J^o(\rho\gamma^{-1})\|_\tau$$

where $D'_{\gamma_0} := \frac{1}{2}\log\|\rho\gamma_0\|_\tau + \frac{1}{2}\log\|\rho\gamma_0^{-1}\|_\tau$. Since $J^o(\rho\gamma)J^o(\rho\gamma^{-1})$ is (r, ε)-proximal, we conclude by applying Lemma B.1.1.

□

Chapter 5

Distribution of the orbit of o with respect to b^o and $b^{o,\tau}$

5.1 Distribution of the orbit of o with respect to b^o

In this section we prove Theorem A. The section is structured as follows: in Subsection 5.1.1 we define a Hölder cocycle over $\partial_\infty\Gamma$ and the corresponding flow. In Subsection 5.1.2 we study the associated Gromov product. Theorem A is proved in Subsection 5.1.4. The contents of Subsection 5.1.3 represent an intermediate step towards the proof of Theorem A.

For the rest of the section we fix $\rho : \Gamma \to \mathrm{G}$ a $\mathrm{P}_1^{p,q}$-Anosov representation and a point o in $\mathbf{\Omega}_\rho$.

5.1.1 The cocycle c_o

Observe that by definition of $\mathbf{\Omega}_\rho$ and equivariance of the curves ξ and η the following map is well defined. Let

$$c_o : \Gamma \times \partial_\infty\Gamma \to \mathbb{R} : \quad c_o(\gamma, x) := \frac{1}{2}\log\left|\frac{\vartheta_x\left((\rho\gamma^{-1})J^o(\rho\gamma)\cdot v_x\right)}{\vartheta_x\left(J^o \cdot v_x\right)}\right|,$$

where $\vartheta_x : \mathbb{R}^{p,q} \to \mathbb{R}$ is a non zero linear functional whose kernel equals $\eta(x)$ and $v_x \neq 0$ belongs to the line $\xi(x)$.

The proof of the following is straightforward.

Lemma 5.1.1. *The map c_o is a Hölder cocycle. The period of $\gamma \in \Gamma_{\mathrm{H}}$ is given by*

$$p_{c_o}(\gamma) = \lambda_1(\rho\gamma) > 0.$$

□

Remark 5.1.2.

- The cocycle c_o admits the following geometric interpretation. One can prove that for every $\gamma \in \Gamma$ and $x \in \partial_\infty \Gamma$ one has

$$c_o(\gamma, x) = \beta_{\xi(x)}(\rho\gamma^{-1} \cdot o, o)$$

where $\beta.(\cdot,\cdot)$ is the pseudo-Riemannian Busemann function defined by Glorieux-Monclair [23, Definition 3.8]. This description will not be used in the future.

- Let τ be a point in S^o. The cocycle c_o is cohomologous to the projective τ-Busemann cocycle of ρ (Subsection 2.3.2). Explicitly, the function

$$V : \partial_\infty \Gamma \to \mathbb{R} : \quad V(x) := \frac{1}{2} \log \frac{\|v_x\|_\tau^2}{|\langle v_x, J^o \cdot v_x \rangle_{p,q}|},$$

where v_x is any non zero vector in the line $\xi(x)$, gives the cohomological relation

$$c_\tau^1(\gamma, x) - c_o(\gamma, x) = V(\gamma \cdot x) - V(x)$$

for every $(\gamma, x) \in \Gamma \times \partial_\infty \Gamma$.

◇

Denote by $\mathrm{U}_o\Gamma$ the quotient space of $\tilde{\mathrm{U}\Gamma}$ under the action of Γ induced by c_o. By Remark 5.1.2 the translation flow ψ_t is Hölder conjugate to the translation flow on $\mathrm{U}_\tau^1\Gamma$. In particular, the topological entropy of ψ_t coincides with the entropy h_ρ^1 of ρ.

Remark 5.1.3. One can prove that if we "push" this construction by the limit map $\xi : \partial_\infty \Gamma \to \mathrm{L}_\rho^1$ we recover the geodesic flow defined in [23, Subsection 6.1] for $\mathsf{H}^{p,q-1}$-convex co-compact groups. This remark will not be used in the sequel.

◇

5.1.2 Dual cocycle and Gromov product

The cocycle c_o is dual to itself, i.e. $p_{c_o}(\gamma) = p_{c_o}(\gamma^{-1})$ for every $\gamma \in \Gamma_\mathrm{H}$. Indeed, this follows from Lemma 5.1.1 and the fact that $\lambda_1(g) = \lambda_1(g^{-1})$ for all g in G. In particular, c_o is cohomologous to the cocycle $\bar{c}_\tau^1$ of Subsection 2.3.2.

Thanks to transversality condition (2.1.2) and the fact that o belongs to $\mathbf{\Omega}_\rho$ the following map is well defined. Let

$$[\cdot,\cdot]_o : \partial^2_\infty\Gamma \to \mathbb{R} : \quad [x,y]_o := -\frac{1}{2}\log\left|\frac{\vartheta_x\left(J^o\cdot v_x\right)\vartheta_y\left(J^o\cdot v_y\right)}{\vartheta_x\left(v_y\right)\vartheta_y\left(v_x\right)}\right|,$$

where ϑ_x (resp. ϑ_y) is a non zero linear functional whose kernel is $\eta(x)$ (resp. $\eta(y)$) and v_x (resp. v_y) is a non zero vector in the line $\xi(x)$ (resp. $\xi(y)$). The proof of the following lemma is a direct computation.

Lemma 5.1.4. *The map $[\cdot,\cdot]_o$ is a Gromov product for the pair (c_o, c_o).*

□

Remark 5.1.5. The map $[\cdot,\cdot]_o$ coincides, up to a sign, with the Gromov product introduced in [23, Subsection 3.5]. The authors give geometric interpretations of this function using pseudo-Riemannian geometry.

◇

The following lemma will be very important in the proof of Theorem A. It provides a geometric interpretation of the Gromov product different from the one given in Remark 5.1.5.

Lemma 5.1.6. *Let γ be an element of Γ_{H}. Then*

$$[\gamma_-,\gamma_+]_o = -\tfrac{1}{2}\mathbb{B}^1(J^o\cdot(\rho\gamma)_-, J^o\cdot(\rho\gamma)_+, (\rho\gamma^{-1})_-, (\rho\gamma^{-1})_+).$$

Proof. Recall that $\rho\gamma^{\pm 1}$ is proximal and that the following holds:

$$(\rho\gamma)_+ = \xi(\gamma_+), \quad (\rho\gamma^{-1})_+ = \xi(\gamma_-), \quad (\rho\gamma)_- = \eta(\gamma_-), \quad (\rho\gamma^{-1})_- = \eta(\gamma_+).$$

Since $J^o = (J^o)^{-1}$, the matrix $J^o(\rho\gamma)J^o$ is proximal and one has the equalities

$$(J^o(\rho\gamma)J^o)_+ = J^o\cdot\xi(\gamma_+) \text{ and } (J^o(\rho\gamma)J^o)_- = J^o\cdot\eta(\gamma_-).$$

The proof finishes by a direct computation.

□

5.1.3 Distribution of attractors and repellors with respect to b^o

By Remark 2.3.1 and Proposition 2.3.6 there exists a Patterson-Sullivan probability (of dimension $h := h^1_\rho$) associated to c_o. From Proposition 2.3.7 and Remark 2.3.8 one has the following.

Proposition 5.1.7 (Sambarino [60, Proposition 4.3])**.** *There exists a constant $\mathtt{m}' = \mathtt{m}'_{\rho,o} > 0$ such that*

$$\mathtt{m}' e^{-ht}\sum_{\gamma\in\Gamma_{\mathrm{H}},\lambda_1(\rho\gamma)\le t}\delta_{\gamma_-}\otimes\delta_{\gamma_+} \to e^{-h[\cdot,\cdot]_o}\mu_o\otimes\mu_o$$

as $t \to \infty$ on $C_c^*(\partial_\infty^2 \Gamma)$.

□

Fix a point $\tau \in \mathsf{S}^o$, a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ and a closed Weyl chamber $\mathfrak{b}^+$ contained in $\mathfrak{b}$. The following proposition is an intermediate step towards the proof of Theorem A. It implies

$$\#\{\gamma \in \Gamma_{\mathrm{H}} : \quad \rho\gamma \in \mathscr{C}_{o,\mathrm{G}}^{>} \text{ and } \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t\} \sim \frac{e^{ht}}{\mathtt{m}'}.$$

In order to obtain Theorem A we still need to count torsion elements γ for which $\|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t$. This counting will be a consequence of Proposition 5.1.8.

Proposition 5.1.8. *There exists a constant $\mathtt{m}' = \mathtt{m}'_{\rho,o} > 0$ such that*

$$\mathtt{m}' e^{-ht} \sum_{\gamma \in \Gamma_{\mathrm{H}}, \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \to \mu_o \otimes \mu_o$$

as $t \to \infty$ on $C^*(\partial_\infty \Gamma \times \partial_\infty \Gamma)$.

Recall that the projection b^o is only defined in the set $\mathscr{C}_{o,\mathrm{G}}^{>}$. The sum in Proposition 5.1.8 is taken then over all elements $\gamma \in \Gamma_{\mathrm{H}}$ for which $\rho\gamma \in \mathscr{C}_{o,\mathrm{G}}^{>}$ and $\|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t$. To make the formula more readable we do not emphasize this. On the other hand, by Corollary 4.2.2 this condition holds except for finitely many exceptions $\gamma \in \Gamma$.

Proof of Proposition 5.1.8. Set

$$\theta_t := \mathtt{m}' e^{-ht} \sum_{\gamma \in \Gamma_{\mathrm{H}}, \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+}.$$

We first prove the statement outside the diagonal, that is, on subsets of $\partial_\infty^2 \Gamma$. Let $\delta > 0$ and $A, B \subset \partial_\infty \Gamma$ disjoint open sets. Consider an element $\gamma \in \Gamma_{\mathrm{H}}$ such that $\gamma_- \in A$ and $\gamma_+ \in B$ and let $s := [\gamma_-, \gamma_+]_o$. By taking A and B smaller we may assume

$$|[x, y]_o - s| < \delta \tag{5.1.1}$$

for all $(x, y) \in A \times B$.

By Lemma 4.3.3, apart from possibly finitely many exceptions $\gamma \in \Gamma_{\mathrm{H}}$ with $(\gamma_-, \gamma_+) \in A \times B$, the following holds:

$$\left| \|b^o(\rho\gamma)\|_{\mathfrak{b}} - \lambda_1(\rho\gamma) - \tfrac{1}{2}\mathbb{B}^1(J^o \cdot (\rho\gamma)_-, J^o \cdot (\rho\gamma)_+, (\rho\gamma^{-1})_-, (\rho\gamma^{-1})_+) \right| < \delta.$$

Applying Lemma 5.1.6 we conclude that

$$|\|b^o(\rho\gamma)\|_{\mathfrak{b}} - \lambda_1(\rho\gamma) + [\gamma_-, \gamma_+]_o| < \delta.$$

By equation (5.1.1) it follows that

$$\lambda_1(\rho\gamma) - s - 2\delta < \|b^o(\rho\gamma)\|_{\mathfrak{b}} < \lambda_1(\rho\gamma) - s + 2\delta$$

holds apart from finitely many exceptions $\gamma \in \Gamma_{\mathrm{H}}$ such that $\gamma_- \in A$ and $\gamma_+ \in B$. From now on, the proof of the convergence

$$\theta_t(A \times B) \to \mu_o(A)\mu_o(B)$$

follows line by line the proof of [60, Theorem 6.5].

It remains to prove the convergence on the diagonal $\{(x,x) : \quad x \in \partial_\infty\Gamma\}$, but once again, the proof is the same as the one given in [60, Theorem 6.5]. For completeness we briefly sketch it.

Since μ_o has no atoms (c.f. Lemma 2.3.4), for every γ in Γ the diagonal has $(\mu_o \otimes \gamma_*\mu_o)$-measure equal to zero. We fix two elements $\gamma_0, \gamma_1 \in \Gamma_{\mathrm{H}}$ with no common fixed point in $\partial_\infty\Gamma$ and let $\varepsilon_0 > 0$. There exists a finite open covering $\mathscr{U}$ of $\partial_\infty\Gamma$ such that for $i = 0, 1$ one has

$$\sum_{U \in \mathscr{U}} \mu_o(U)\mu_o(\gamma_i^{-1} \cdot U) < \varepsilon_0.$$

We can assume that for every $U \in \mathscr{U}$ there exists $i \in \{0,1\}$ such that $\gamma_i^{-1} \cdot \overline{U}$ is disjoint from $\overline{U}$. There exists an open covering $\mathscr{V}$ of $\partial_\infty\Gamma$ with the following properties:

1. $\sum_{V \in \mathscr{V}} \mu_o(V)\mu_o(\gamma_i^{-1} \cdot V) < \varepsilon_0$ for $i = 0, 1$.

2. The closure of every element in $\mathscr{U}$ is contained in a unique element of $\mathscr{V}$ and if $\gamma_i^{-1} \cdot \overline{U}$ is disjoint from $\overline{U}$ the same holds for this element in $\mathscr{V}$.

3. Suppose that $\gamma_i^{-1} \cdot \overline{U} \cap \overline{U} = \emptyset$ and let $V \in \mathscr{V}$ be the unique element such that $\overline{U} \subset V$. Then apart from finitely many exceptions γ such that $\gamma_\pm \in U$ one has $(\gamma_i^{-1}\gamma)_- \in V$ and $(\gamma_i^{-1}\gamma)_+ \in \gamma_i^{-1} \cdot V$.

Set $D := \max_{i=0,1}\{D_{\gamma_i^{-1}}\}$ where $D_{\gamma_i^{-1}}$ is the constant given by Proposition 4.5.1 and take $U \in \mathscr{U}$ as in item 3. above. By Proposition 4.5.1 we have

$$\begin{aligned}\theta_t(U \times U) \leq \mathtt{m}' e^{-ht} & \sum_{\gamma \in \Gamma_{\mathrm{H}}, \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t + D} \delta_{\gamma_-}(V)\delta_{\gamma_+}(\gamma_i^{-1} \cdot V) \\ & + \mathtt{m}' e^{-ht} \# F\end{aligned}$$

where F is a finite set independent of t. Since $V \times \gamma_i^{-1} \cdot V$ is far from the diagonal the right side converges to

$$e^D \mu_o(V)\mu_o(\gamma_i^{-1} \cdot V)$$

as $t \to \infty$. Adding up in $U \in \mathscr{U}$ we conclude

$$\limsup_{t\to\infty} \sum_{U \in \mathscr{U}} \theta_t(U \times U) \leq 2e^D \varepsilon_0.$$

Hence $\theta_t(\{(x,x) : \quad x \in \partial_\infty \Gamma\})$ converges to zero and since the diagonal has measure zero for $\mu_o \otimes \mu_o$ the proof is finished.

□

5.1.4 Proof of Theorem A

The following is a corollary of Proposition 5.1.8.

Corollary 5.1.9. *There exists a constant* $\mathtt{m}' = \mathtt{m}'_{\rho,o} > 0$ *such that*

$$\mathtt{m}' e^{-ht} \sum_{\gamma \in \Gamma_{\mathrm{H}}, \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t} \delta_{\rho\gamma^{-1} \cdot o^{\perp_{p,q}}} \otimes \delta_{\rho\gamma \cdot o} \to \eta_*(\mu_o) \otimes \xi_*(\mu_o)$$

on $C^*(\mathsf{P}((\mathbb{R}^{p,q})^*) \times \mathsf{P}(\mathbb{R}^{p,q}))$ *as* $t \to \infty$.

Proof. Set

$$\nu_t^{\mathrm{H}} := \mathtt{m}' e^{-ht} \sum_{\gamma \in \Gamma_{\mathrm{H}}, \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t} \delta_{\rho\gamma^{-1} \cdot o^{\perp_{p,q}}} \otimes \delta_{\rho\gamma \cdot o}$$

and take θ_t the measure defined in the proof of Proposition 5.1.8. We know that

$$(\eta, \xi)_*(\theta_t) \to \eta_*(\mu_o) \otimes \xi_*(\mu_o).$$

Hence we only have to show the following convergence

$$\nu_t^{\mathrm{H}} - (\eta, \xi)_*(\theta_t) \to 0. \tag{5.1.2}$$

Take a small positive δ. By Proposition 2.1.3 and the proof of Proposition 4.2.1 we know that, apart from finitely many exceptions γ in Γ_{H}, one has

$$d(\rho\gamma \cdot o, (\rho\gamma)_+) < \delta \text{ and } d(\rho\gamma^{-1} \cdot o, (\rho\gamma^{-1})_+) < \delta.$$

By taking $\cdot^{\perp_{p,q}}$ we can assume further that[1]

$$d^*(\rho\gamma^{-1} \cdot o^{\perp_{p,q}}, (\rho\gamma)_-) < \delta.$$

Now the proof of the convergence of equation (5.1.2) follows from evaluation on continuous functions of $\mathsf{P}((\mathbb{R}^{p,q})^*) \times \mathsf{P}(\mathbb{R}^{p,q})$.

□

[1] Recall that $d^*(\cdot,\cdot)$ denotes the distance in $\mathsf{P}((\mathbb{R}^{p,q})^*)$ induced by the operator norm.

We finally finish the proof of Theorem A.

Proposition 5.1.10. *There exists a constant* $\mathtt{m}' = \mathtt{m}'_{\rho,o} > 0$ *such that*

$$\mathtt{m}' e^{-ht} \sum_{\gamma\in\Gamma, \|b^o(\rho\gamma)\|_{\mathfrak{b}}\leq t} \delta_{\rho\gamma^{-1}\cdot o^{\perp_{p,q}}} \otimes \delta_{\rho\gamma\cdot o} \to \eta_*(\mu_o)\otimes\xi_*(\mu_o)$$

on $C^*(\mathsf{P}((\mathbb{R}^{p,q})^*)\times\mathsf{P}(\mathbb{R}^{p,q}))$ *as* $t\to\infty$*.*

Proof. The structure of the proof is the same as that of Proposition 5.1.8, that is, we first prove the statement outside the diagonal and deduce from that the statement on the diagonal. Here by *diagonal* we mean the set

$$\mathscr{D} := \{(\vartheta, v) \in \mathsf{P}((\mathbb{R}^{p,q})^*)\times\mathsf{P}(\mathbb{R}^{p,q}) : \quad \vartheta(v) = 0\}.$$

Let

$$\nu_t := \mathtt{m}' e^{-ht} \sum_{\gamma\in\Gamma, \|b^o(\rho\gamma)\|_{\mathfrak{b}}\leq t} \delta_{\rho\gamma^{-1}\cdot o^{\perp_{p,q}}} \otimes \delta_{\rho\gamma\cdot o}$$

and take ν_t^{H} as in the proof of Corollary 5.1.9.

Consider first a continuous function f on $\mathsf{P}((\mathbb{R}^{p,q})^*)\times\mathsf{P}(\mathbb{R}^{p,q})$ whose support $\mathrm{supp}(f)$ is disjoint from $\mathscr{D}$.

Claim 5.1.11. The following holds

$$\#\{\gamma\in\Gamma : \quad (\rho\gamma^{-1}\cdot o^{\perp_{p,q}}, \rho\gamma\cdot o) \in \mathrm{supp}(f) \text{ and } \gamma\notin\Gamma_{\mathrm{H}}\} < \infty.$$

Proof of Claim 5.1.11. Fix a positive D such that for every $(\vartheta, v)\in\mathrm{supp}(f)$ one has $d(\vartheta, v) > D$. As we saw in the proof of Proposition 4.2.1, the distances

$$d(\rho\gamma\cdot o, U_1^\tau(\rho\gamma)) \text{ and } d^*(\rho\gamma^{-1}\cdot o^{\perp_{p,q}}, S_{d-1}^\tau(\rho\gamma))$$

converge to zero as $\gamma\to\infty$. We conclude that, apart from possibly finitely many exceptions γ in Γ with $(\rho\gamma^{-1}\cdot o^{\perp_{p,q}}, \rho\gamma\cdot o)\in\mathrm{supp}(f)$, one has

$$d(U_1^\tau(\rho\gamma), S_{d-1}^\tau(\rho\gamma)) > D.$$

Now apply equation (2.1.1), Remark 1.5.2 and Benoist's Lemma B.1.2 to conclude that for $|\gamma|_\Gamma$ large enough the matrix $\rho\gamma$ is proximal.

□

From Claim 5.1.11 we conclude that

$$\lim_{t\to\infty}\nu_t(f) = \lim_{t\to\infty}\nu_t^{\mathrm{H}}(f)$$

which by Corollary 5.1.9 equals $(\eta_*(\mu_o)\otimes\xi_*(\mu_o))(f)$.

It remains to prove the convergence on the diagonal. It suffices to prove that for every positive ε_0 there exists an open covering $\{U^*\times U\}$ of $\mathscr{D}$ such that

$$\limsup_{t\to\infty}\nu_t\left(\bigcup(U^*\times U)\right)\leq\varepsilon_0.$$

The proof is the same as that of Proposition 5.1.8. Namely, take two elements γ_0,γ_1 in Γ_{H} with no common fixed point in $\partial_\infty\Gamma$ and coverings $\mathscr{U}=\{U^*\times U\}$ and $\mathscr{V}=\{V^*\times V\}$ of $\mathscr{D}$ by open sets with the following properties:

1. For every $U^*\times U$ in $\mathscr{U}$ there exists $i=0,1$ such that $\rho\gamma_i^{-1}\cdot\overline{U}$ is transverse to $\overline{U^*}$.
2. $\displaystyle\sum_{V^*\times V\in\mathscr{V}}(\eta_*(\mu_o)\otimes\xi_*(\mu_o))(V^*\times\rho\gamma_i^{-1}\cdot V)<\varepsilon_0$ for $i=0,1$.
3. The closure of every element in $\mathscr{U}$ is contained in a unique element of $\mathscr{V}$ and if $\rho\gamma_i^{-1}\cdot\overline{U}$ is transverse to $\overline{U^*}$ the same holds for this element in $\mathscr{V}$.
4. Suppose that $\rho\gamma_i^{-1}\cdot\overline{U}$ is transverse to $\overline{U^*}$ and let $V^*\times V\in\mathscr{V}$ be the unique element such that $\overline{U}\subset V$ and $\overline{U^*}\subset V^*$. Then, apart from possibly finitely many exceptions γ such that $(\rho\gamma^{-1}\cdot o^{\perp_{p,q}},\rho\gamma\cdot o)\in U^*\times U$, one has
$$((\rho\gamma_i^{-1})(\rho\gamma^{-1})\cdot o^{\perp_{p,q}},(\rho\gamma_i^{-1})(\rho\gamma)\cdot o)\in V^*\times\rho\gamma_i^{-1}\cdot V.$$

Provided with this construction, the proof finishes in the same way as that of Proposition 5.1.8.

□

Remark 5.1.12. From Proposition 5.1.10 we deduce that

$$\lim_{t\to\infty}\frac{\log\#\{\gamma\in\Gamma:\quad\rho\gamma\in\mathscr{C}_{o,\mathrm{G}}^{>}\text{ and }\|b^o(\rho\gamma)\|_{\mathfrak{b}}\leq t\}}{t}$$

coincides with the entropy $h=h_\rho^1$ of ρ.

◇

5.2 Distribution of the orbit of o with respect to $b^{o,\tau}$

The proof of Theorem B follows the same lines of the proof of Theorem A, we just have to pick a (slightly) different flow ψ_t.

Fix a $\mathrm{P}_1^{p,q}$-Anosov representation $\rho:\Gamma\to\mathrm{G}$, a point o in $\mathbf{\Omega}_\rho$ and $\tau\in\mathsf{S}^o$.

5.2.1 The cocycle c_τ

Let $\|\cdot\|_\tau$ be the norm on $\mathbb{R}^{p,q}$ associated to τ and $\|\cdot\|^*_\tau$ be the induced norm on $(\mathbb{R}^{p,q})^*$. Let

$$c_\tau : \Gamma \times \partial_\infty\Gamma \to \mathbb{R} : \quad c_\tau(\gamma, x) := \frac{1}{2}\log\left(\frac{\|\rho\gamma \cdot \vartheta_x\|^*_\tau \|\rho\gamma \cdot v_x\|_\tau}{\|\vartheta_x\|^*_\tau \|v_x\|_\tau}\right)$$

where $\vartheta_x : \mathbb{R}^{p,q} \to \mathbb{R}$ is a non zero linear functional whose kernel equals $\eta(x)$ and v_x is a non zero vector in the line $\xi(x)$.

Since $\lambda_1(g) = \lambda_1(g^{-1})$ for every $g \in \mathrm{G}$, one can prove that the following equalities hold

$$c_\tau = c^1_\tau = \bar{c}^1_\tau,$$

where c^1_τ and $\bar{c}^1_\tau$ are the projective τ-Busemann cocycles of ρ (Section 2.3.2).

5.2.2 Dual cocycle and Gromov product

Recall that c_o is the cocycle defined in Section 5.1. The cocycle c_τ is dual to c_o, i.e. $p_{c_o}(\gamma) = p_{c_\tau}(\gamma^{-1})$ for every $\gamma \in \Gamma_{\mathrm{H}}$. Let

$$[\cdot,\cdot]_{o,\tau} : \partial^2_\infty\Gamma \to \mathbb{R} : \quad [x,y]_{o,\tau} := \frac{1}{2}\log\left|\frac{\vartheta_y(v_x)\,\vartheta_x(v_y)}{\vartheta_x(J^o \cdot v_x)\,\|\vartheta_y\|^*_\tau \|v_y\|_\tau}\right|.$$

The proof of the following lemma is a direct computation.

Lemma 5.2.1. *The map $[\cdot,\cdot]_{o,\tau}$ is a Gromov product for the pair (c_o, c_τ).*

□

Recall that $\mathbb{G}^1_\tau$ is the projective τ-Gromov product (see Subsection B.1.1).

Lemma 5.2.2. *Let γ be an element of Γ_{H}. Then the number $[\gamma_-, \gamma_+]_{o,\tau}$ coincides with*

$$-\tfrac{1}{2}\mathbb{B}^1(J^o \cdot (\rho\gamma)_-, J^o \cdot (\rho\gamma)_+, (\rho\gamma^{-1})_-, (\rho\gamma^{-1})_+) + \tfrac{1}{2}\mathbb{G}^1_\tau((\rho\gamma^{-1})_-, J^o \cdot (\rho\gamma)_+).$$

Proof. Recall the definition of $[\cdot,\cdot]_o$ from Subsection 5.1.2. One has

$$[\gamma_-, \gamma_+]_{o,\tau} = [\gamma_-, \gamma_+]_o + \frac{1}{2}\log\frac{|\vartheta_{\gamma_+}(J^o \cdot v_{\gamma_+})|}{\|\vartheta_{\gamma_+}\|^*_\tau \|v_{\gamma_+}\|_\tau}.$$

The proof then follows from Lemma 5.1.6 and Remark 1.2.5.

□

5.2.3 Distribution of attractors and repellors with respect to $b^{o,\tau}$

Let μ_τ be a Patterson-Sullivan probability (of dimension $h = h^1_\rho$) for the cocycle c_τ and recall that μ_o is the one associated to c_o. By Remark 2.3.8, the Bowen-Margulis measure of the flow ψ_t on $\mathrm{U}^1_\tau\Gamma$ admits the following disintegration

$$e^{-h[\cdot,\cdot]_{o,\tau}}\mu_o \otimes \mu_\tau \otimes dt.$$

Let $\mathfrak{b}^+$ be a closed Weyl chamber of a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$.

Proposition 5.2.3. *There exists a constant* $\mathtt{m}'' = \mathtt{m}''_{\rho,o,\tau} > 0$ *such that*

$$\mathtt{m}'' e^{-ht} \sum_{\gamma \in \Gamma_{\mathrm{H}}, \|b^{o,\tau}(\rho\gamma)\|_{\mathfrak{b}} \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \to \mu_o \otimes \mu_\tau$$

as $t \to \infty$ *on* $C^*(\partial_\infty\Gamma \times \partial_\infty\Gamma)$.

Proof. The proof is the same that the one given in Proposition 5.1.8 adapted to the pair (c_o, c_τ) and the Gromov product $[\cdot,\cdot]_{o,\tau}$: apply item *5.* of Lemma 4.3.3 and Lemma 5.2.2.

□

5.2.4 Proof of Theorem B

Provided Proposition 5.2.3, the following proposition can be proved in the same way as Proposition 5.1.10.

Proposition 5.2.4. *There exists a constant* $\mathtt{m}'' = \mathtt{m}''_{\rho,o,\tau} > 0$ *such that*

$$\mathtt{m}'' e^{-ht} \sum_{\gamma \in \Gamma, \|b^{o,\tau}(\rho\gamma)\|_{\mathfrak{b}} \leq t} \delta_{\rho\gamma^{-1}\cdot o^{\perp_{p,q}}} \otimes \delta_{\rho\gamma\cdot o} \to \eta_*(\mu_o) \otimes \xi_*(\mu_\tau)$$

on $C^*(\mathsf{P}((\mathbb{R}^{p,q})^*) \times \mathsf{P}(\mathbb{R}^{p,q}))$ *as* $t \to \infty$.

□

Part III

Counting for $\mathrm{G} = \mathrm{PSL}(V)$ and $\mathrm{H}^o \cong \mathrm{PSO}(p, q)$

Throughout this part of the thesis G denotes the Lie group $\mathrm{PSL}(V)$ for some real vector space V of dimension $d \geq 2$. We now initiate the study of some quantitative geometric aspects of the pseudo-Riemannian symmetric space $\mathsf{Q}_{p,q}$ of (homothety classes of) quadratic forms of signature (p,q) and provide, as in the previous part, an interpretation of these quantities in the context of the Riemannian symmetric space X_{G}. Notably, for a given basepoint $o \in \mathsf{Q}_{p,q}$ we introduce the subset

$$\mathscr{B}_{o,\mathrm{G}}$$

consisting on elements of G that take some o-orthogonal basis of lines of V into another o-orthogonal basis of lines. We then show (Proposition 7.1.1) the analogue of Proposition 3.2.1 in the present setting: we have a (p,q)-*Cartan decomposition*

$$\mathscr{B}_{o,\mathrm{G}} = \mathrm{H}^o \mathrm{W} \exp(\mathfrak{b}^+) \mathrm{H}^o$$

of $\mathscr{B}_{o,\mathrm{G}}$. We define the associated (p,q)-*Cartan projection*

$$b^o : \mathscr{B}_{o,\mathrm{G}} \to \mathfrak{b}^+.$$

Geometric interpretations of this projection are discussed in Section 7.3.

We then begin the study of the (p,q)-Cartan projection. Recall that in this case we can take $\mathfrak{b} = \mathfrak{a}$, that is, the maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ is also maximal in $\mathfrak{p}^\tau$ (Example 1.2.6). Therefore $\mathfrak{b}^+$ is a union of $\mathfrak{b}^+$-*compatible Weyl chambers*, i.e. Weyl chambers of the system $\Sigma(\mathfrak{g},\mathfrak{a})$ which are contained in $\mathfrak{b}^+$. In Proposition 7.2.3 we actually compute the $\mathfrak{b}^+$-compatible Weyl chamber that contains $b^o(g)$ under the assumptions that g has a sufficiently strong gap of index Δ and a "genericity condition" for its Cartan attractor and repellor (they must belong to $\mathsf{F}(V)^o$, the union of open orbits of the action $\mathrm{H}^o \curvearrowright \mathsf{F}(V)$). This makes the study of the (p,q)-Cartan projection tractable. For instance in Section 7.3.3 we show that under these assumptions one has

$$b^o(g) = \frac{1}{2} w_g \cdot \lambda(\sigma^o(g^{-1})g) \text{ and } \|b^o(g) - w_g \cdot a^\tau(g)\| \leq D \tag{5.2.1}$$

for some positive constant D and an element w_g of the Weyl group that we can precisely describe. Whenever a Borel-Anosov representation

$$\rho : \Gamma \to \mathrm{G}$$

and a basepoint o in the set

$$\mathbf{\Omega}_\rho := \{o \in \mathsf{Q}_{p,q} : \quad \xi_\rho(x) \in \mathsf{F}(V)^o \text{ for all } x \in \partial_\infty \Gamma\}$$

are given, these results are sufficient to study the *asymptotic cone* of $b^o(\rho(\Gamma))$ and its *growth indicator*, by means of Benoist's asymptotic cone [4] and Quint's growth indicator [55] (see Subsection 9.1.3 for further precisions). Also, we show in Corollary 9.1.3 that the (p,q)-*critical exponent*

$$\limsup_{t\to\infty} \frac{\log \# \{\gamma \in \Gamma : \quad \rho\gamma \in \mathscr{B}_{o,\mathrm{G}} \text{ and } \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t\}}{t}$$

of ρ is positive, finite and independent on the choice of the basepoint $o \in \mathbf{\Omega}_\rho$. Let us emphasize however that equalities (5.2.1) strongly depend on the choice of a $\mathfrak{b}^+$-compatible Weyl chamber (which is needed to define the projections λ and a^τ) and that there seems to be no canonical way to choose this Weyl chamber.

In Chapter 8 we introduce an *o-Busemann cocycle* and a corresponding *o-Gromov product* for $\mathscr{B}_{o,\mathrm{G}}$ which are, as in the Riemannian setting, the key objects for the study of more precise asymptotic properties of the (p,q)-Cartan projection. Again these objects seem to be not canonical: their definition depends on the choice of a $\mathfrak{b}^+$-compatible Weyl chamber. In Section 9.2 we prove our asymptotic results for $b^o(\rho(\Gamma))$ (Proposition 9.2.10) under the assumption that the limit set $\xi(\partial_\infty\Gamma)$ of ρ is contained in a single open orbit of the action

$$\mathrm{H}^o \curvearrowright \mathsf{F}(V).$$

As we will see, this assumption selects a $\mathfrak{b}^+$-compatible Weyl chamber and therefore an o-Busemann cocycle for ρ can be naturally defined.

Chapter 6

Preliminaries

We begin with some geometric and linear algebraic considerations in our framework. In Section 6.1 we treat with flat subspaces of X_{G} and $\mathsf{Q}_{p,q}$: these considerations will allow us to discuss geometric interpretations of the (p,q)-Cartan projection in both settings. Sections 6.2 and 6.3 are mainly technical and intended to introduce some terminology that will be used throughout this part of the thesis. The most relevant section of the present chapter is Section 6.4, in which we study the set of open orbits of the action of H^o on $\mathsf{F}(V)$.

6.1 Flat submanifolds

6.1.1 Flats in X_{G} orthogonal to S^o

A *flat* in X_{G} is a maximal dimensional totally geodesic submanifold of X_{G} on which sectional curvature vanishes. Let τ be a point in X_{G}. The set of flats in X_{G} (through the basepoint τ) is in one to one correspondence with the set of Cartan subspaces of $\mathfrak{g}$ (contained in $\mathfrak{p}^\tau$). Further, the choice of a Cartan subspace of $\mathfrak{g}$ (contained in $\mathfrak{p}^\tau$) is equivalent to the choice of a (τ-orthogonal) basis of lines of V (recall Remark 1.2.7). More concretely, to a basis of lines $\mathcal{C}$ one associates the flat

$$\{\tau' \in \mathsf{X}_{\mathrm{G}} : \quad \mathcal{C} \text{ is } \tau'\text{-orthogonal}\}.$$

If $\mathcal{C}$ is τ-orthogonal and $\mathfrak{a} \subset \mathfrak{p}^\tau$ is the Cartan subspace associated to $\mathcal{C}$, this flat coincides with

$$\exp(\mathfrak{a}) \cdot \tau.$$

From Example 1.2.6 and the contents of Section 1.3 we deduce the following.

Proposition 6.1.1. *Let $o \in \mathsf{Q}_{p,q}$, τ be a point in S^o and $\mathfrak{a} \subset \mathfrak{p}^\tau$ be a maximal subalgebra. Then the following are equivalent:*

1. *The flat* $\exp(\mathfrak{a}) \cdot \tau$ *is orthogonal to* S^o *at* τ.

2. *The inclusion* $\mathfrak{a} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ *holds.*

3. *The basis of lines* $\mathcal{C}$ *associated to* $\mathfrak{a}$ *is* o*-orthogonal (and* τ*-orthogonal).*

□

6.1.2 Space-like flats in $\mathsf{Q}_{p,q}$

In contrast with the Riemannian case, the choice of a Cartan subspace in $\mathfrak{g}$ does not determine a flat in $\mathsf{Q}_{p,q}$, but rather a disjoint union of flats. To see this, let $\mathcal{C}$ be a basis of lines of V. In analogy with the Riemannian setting, it is natural to consider the subset

$$\{o \in \mathsf{Q}_{p,q} : \quad \mathcal{C} \text{ is } o\text{-orthogonal}\} \tag{6.1.1}$$

of $\mathsf{Q}_{p,q}$ associated to this choice. Let

$$\Upsilon_{p,q} := \{\upsilon : \mathcal{C} \to \{-1, 1\} : \quad \#\{\ell : \upsilon(\ell) = 1\} = p\}.$$

Each element $\upsilon \in \Upsilon_{p,q}$ determines the subset of $\mathsf{Q}_{p,q}$ given by

$$[f_\upsilon] := \{o \in \mathsf{Q}_{p,q} : \quad \mathcal{C} \text{ is } o\text{-orthogonal and } \mathrm{sg}_o(\ell) = \upsilon(\ell) \;\; \forall \ell \in \mathcal{C}\}$$

and the set (6.1.1) coincides with the (disjoint) union of the sets $[f_\upsilon]$ as υ ranges over $\Upsilon_{p,q}$. The choice of a point

$$o \in \bigsqcup_{\upsilon \in \Upsilon_{p,q}} [f_\upsilon]$$

determines an element $\upsilon_o \in \Upsilon_{p,q}$ characterized by the equalities

$$\upsilon_o(\ell) = \mathrm{sg}_o(\ell)$$

for every $\ell \in \mathcal{C}$. Further, if $\mathfrak{b}$ is the Cartan subspace of $\mathfrak{g}$ associated to $\mathcal{C}$ then one has $\mathfrak{b} \subset \mathfrak{q}^o$ and

$$\exp(\mathfrak{b}) \cdot o = [f_{\upsilon_o}].$$

Because of Example 1.2.6 we can take an element $\tau \in \mathsf{S}^o$ for which one has further $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ and we conclude that the set (6.1.1) is a disjoint union of space-like flats in $\mathsf{Q}_{p,q}$.

6.2 Weyl groups and Weyl chambers

6.2.1 Geometric interpretation of Weyl chambers of $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$

We fix for the rest of the chapter a basepoint $o \in \mathsf{Q}_{p,q}$, a Cartan involution $\tau \in \mathsf{S}^o$ and a closed Weyl Chamber $\mathfrak{b}^+$ of the system $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$ for a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$. From Subsection 1.4.1 we recall that if $\mathcal{C}$ the basis of lines of V associated to $\mathfrak{b}$ then the choice of $\mathfrak{b}^+$ corresponds to the choice of a total order in $\mathcal{C}^+$ and in $\mathcal{C}^-$, where $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) is the subset of $\mathcal{C}$ consisting on positive (resp. negative) lines for the form o.

Let $\mathrm{int}(\mathfrak{b}^+)$ be the interior of $\mathfrak{b}^+$.

Lemma 6.2.1. *Let X be an element of $\mathrm{int}(\mathfrak{b}^+)$ and λ be an eigenvalue of X. Then multiplicity of λ is at most two. Moreover, if the multiplicity of λ is equal to two, then the form o restricted to the corresponding eigenspace of X has signature $(1,1)$.*

□

We now provide a geometric interpretation of Weyl chambers for the system $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$.

Proposition 6.2.2. *Let X be an element of $\mathfrak{b}^+$. Then the following are equivalent:*

1. *The basis $\mathcal{C}$ is the unique o-orthogonal and τ-orthogonal basis of lines of V that diagonalizes X.*

2. *The submanifold $\exp(\mathfrak{b}) \cdot \tau$ is the unique flat of X_{G} orthogonal to S^o at τ that contains the geodesic*

$$\exp(\mathbb{R}X) \cdot \tau.$$

3. *The element X belongs to $\mathrm{int}(\mathfrak{b}^+)$.*

Proof. The equivalence *1.*⇔*2.* is a consequence of Proposition 6.1.1. Let us show then the equivalence between *1.* and *3.*

Suppose first that $X \in \mathrm{int}(\mathfrak{b}^+)$. Then one has a unique splitting

$$V = \bigoplus_{i=1}^{j} \ell_i \oplus \bigoplus_{i=1}^{k} \omega_i$$

where $\ell_1, \dots, \ell_j$ (resp. $\omega_1, \dots, \omega_k$) are the eigenspaces of X corresponding to eigenvalues of multiplicity one (resp. two). Further, the signature of o restricted to ω_i is $(1,1)$ for each $i = 1, \dots, k$ and therefore this subspace admits a unique splitting into a direct sum of two lines which is both o-orthogonal and τ-orthogonal. Since $\mathcal{C}$ is o-orthogonal and τ-orthogonal and diagonalizes X we conclude that *1.* must hold.

Conversely, suppose by contradiction that *1.* holds and that X does not belong to $\mathrm{int}(\mathfrak{b}^+)$. Then there exists two lines in $\mathcal{C}$ with the same sign for the form o which are associated to the same eigenvalue of X. Say these two lines are positive for the form o (the negative case being analogous) and let $\tilde{V}$ be the subspace of V spanned by them. We can take representatives $\langle\cdot,\cdot\rangle_\tau$ of τ and $\langle\cdot,\cdot\rangle_o$ of o that coincide on $\tilde{V}$ (c.f. Example 1.3.1). Since any basis of lines of $\tilde{V}$ diagonalizes $X|_{\tilde{V}}$, we find a contradiction.

□

6.2.2 Weyl groups

Recall that $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ is a maximal subalgebra in $\mathfrak{p}^\tau$ and therefore we can denote $\mathfrak{b} = \mathfrak{a}$. Recall also that $\hat{\mathrm{W}} := \mathrm{N}_{\mathrm{K}^\tau}(\mathfrak{a})$ (resp. $\mathrm{M} := \mathrm{Z}_{\mathrm{K}^\tau}(\mathfrak{a})$) is the normalizer (resp. centralizer) of $\mathfrak{a} = \mathfrak{b}$ in K^τ and that

$$\mathrm{W} := \hat{\mathrm{W}}/\mathrm{M}$$

is the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$.

Fix a representative $\langle\cdot,\cdot\rangle_o$ of o[1]. Since the involutions σ^o and σ^τ commute and the basis $\mathcal{C}$ is o-orthogonal and τ-orthogonal, we can find a representative $\langle\cdot,\cdot\rangle_\tau$ of τ for which the following holds: for each line $\ell \in \mathcal{C}$ and each vector $v \in \ell$ one has

$$|\langle v, v\rangle_o| = \langle v, v\rangle_\tau.$$

We deduce the following technical lemma that will be used several times in the future.

Lemma 6.2.3. *Let $\hat{w}$ be an element of G that preserves the set $\mathcal{C}$. Then the following holds:*

1. *If $\hat{w}$ belongs to H^o, then it belongs to K^τ.*

2. *Suppose that $\hat{w}$ belongs to K^τ. Then for every line $\ell \in \mathcal{C}$ and every vector $v \in \ell$ one has*

$$|\langle \hat{w}\cdot v, \hat{w}\cdot v\rangle_o| = |\langle v, v\rangle_o|.$$

In particular, if $\hat{w}$ preserves the o-sign of each line of $\mathcal{C}$ then $\hat{w}$ belongs to H^o.

[1] Recall that by definition this is a symmetric bilinear form associated to a quadratic form in the ray o.

□

By Lemma 6.2.3 we have that M coincides with the centralizer of $\mathfrak{b}$ in H^o and in particular it is contained in H^o. Set

$$[\mathrm{W}] := (\hat{\mathrm{W}} \cap \mathrm{H}^o)\backslash \hat{\mathrm{W}}/\mathrm{M}.$$

Note that [W] is not necessarily a group. We fix from now on the following notations: for an element $\hat{w}$ of $\hat{\mathrm{W}}$ we denote by w its class in W and we set $[w]$ to be the class of this element in [W]. The following is a consequence of Lemma 6.2.3.

Corollary 6.2.4. *Let $\hat{w}_1$ and $\hat{w}_2$ be two elements of $\hat{\mathrm{W}}$. Then the following are equivalent:*

1. *One has the equality $[w_1] = [w_2]$.*
2. *For every $\ell \in \mathcal{C}$ one has*

$$\mathrm{sg}_o(\hat{w}_1 \cdot \ell) = \mathrm{sg}_o(\hat{w}_2 \cdot \ell).$$

□

The following technical lemma will be used repeatedly in Chapter 7.

Lemma 6.2.5. *For every $\hat{w} \in \hat{\mathrm{W}}$ one has that $m := \sigma^o(\hat{w}^{-1})\hat{w}$ belongs to* M*. Moreover, the eigenspace of m associated to the eigenvalue ± 1 coincides with*

$$\mathrm{span}\{\ell \in \mathcal{C} : \quad \mathrm{sg}_o(\hat{w} \cdot \ell) = \pm \mathrm{sg}_o(\ell)\}.$$

Proof. One has that $m = {}^{\star_o}\hat{w}\hat{w}$ belongs to K^τ. Further, for every pair of lines $\ell \neq \ell'$ in $\mathcal{C}$ and vectors $v \in \ell$ and $v' \in \ell'$ one has

$$\langle m \cdot v, v' \rangle_o = \langle \hat{w} \cdot v, \hat{w} \cdot v' \rangle_o = 0.$$

Thus $m \cdot \ell = \ell$ for every $\ell \in \mathcal{C}$ and this shows that m belongs to M. The statement concerning the eigenspaces of m follows from similar arguments.

□

6.2.3 Opposition involution of $\mathfrak{b}^+$

Let $w_{\mathfrak{b}^+} \in \mathrm{W}$ be the unique element that preserves $\mathcal{C}^+$ and $\mathcal{C}^-$ and acts on these sets by reversing the total order induced by $\mathfrak{b}^+$. By Lemma 6.2.3 we have that $\hat{w}_{\mathfrak{b}^+}$ belongs to H^o and, by definition, it satisfies

$$w_{\mathfrak{b}^+} \cdot (-\mathfrak{b}^+) = \mathfrak{b}^+.$$

The *opposition involution* of $\mathfrak{b}^+$ is defined by

$$\iota_{\mathfrak{b}^+} : \mathfrak{b} \to \mathfrak{b} : \quad \iota_{\mathfrak{b}^+}(X) := -w_{\mathfrak{b}^+} \cdot X.$$

Note that $\iota_{\mathfrak{b}^+}$ preserves $\mathfrak{b}^+$.

6.2.4 Compatible Weyl chambers

Recall that $\mathfrak{b} = \mathfrak{a}$ is a Cartan subspace of $\mathfrak{g}$. A $\mathfrak{b}^+$-*compatible Weyl chamber* is a Weyl chamber of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$ that is contained in $\mathfrak{b}^+$. The choice of a $\mathfrak{b}^+$-compatible Weyl chamber corresponds to the choice of a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ that contains $\Sigma^+(\mathfrak{g}^{\tau o}, \mathfrak{b})$. Equivalently, it corresponds to the choice of a total order on $\mathcal{C}$ that in restriction to $\mathcal{C}^{\pm}$ coincides with the total order induced by $\mathfrak{b}^+$.

The following lemma will be important notably in Subsection 6.4.4.

Lemma 6.2.6. *Let $\mathfrak{a}^+ \subset \mathfrak{b}$ be a Weyl chamber of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$. There exists a unique $w \in \mathrm{W}$ such that $\hat{w} \in \mathrm{H}^o$ and*

$$w \cdot \mathfrak{a}^+ \subset \mathfrak{b}^+.$$

Proof. Let $\{\ell_1, \ldots, \ell_d\}$ be total order on $\mathcal{C}$ induced by the choice of $\mathfrak{a}^+$, we define the element w inductively. If $\mathrm{sg}_o(\ell_1) = 1$, we define $w \cdot \ell_1$ to be the first element of $\mathcal{C}^+$ and if $\mathrm{sg}_o(\ell_1) = -1$ we define $w \cdot \ell_1$ to be the first element of $\mathcal{C}^-$. To fix the ideas, let us say that $\mathrm{sg}_o(\ell_1) = 1$. Now we define $w \cdot \ell_2$ to be the first line of $\mathcal{C}^-$ if $\mathrm{sg}_o(\ell_2) = -1$ or the second line of $\mathcal{C}^+$ if not. The inductive process is now clear and defines an element $w \in \mathrm{W}$ for which $w \cdot \mathfrak{a}^+$ is $\mathfrak{b}^+$-compatible. Further, thanks to Lemma 6.2.3 we have $\hat{w} \in \mathrm{H}^o$.

To show uniqueness, observe that the above process is the only one possible.

□

Recall from Section 1.5 that if $\mathfrak{a}^+$ is a Weyl chamber of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$ one has the opposition involution

$$\iota_{\mathfrak{a}^+} : \mathfrak{b} \to \mathfrak{b}$$

of $\mathfrak{a}^+$.

Corollary 6.2.7. *Let $\mathfrak{a}^+$ be a $\mathfrak{b}^+$-compatible Weyl chamber and let $\mathcal{C} = \{\ell_1, \ldots, \ell_d\}$ be the total order on $\mathcal{C}$ induced by this choice. Then the following are equivalent:*

1. *One has the equality $\iota_{\mathfrak{b}^+} = \iota_{\mathfrak{a}^+}$.*
2. *One has the equality $\iota_{\mathfrak{b}^+}(\mathfrak{a}^+) = \mathfrak{a}^+$.*
3. *One has the equality $w_{\mathfrak{b}^+} = w_{\mathfrak{a}^+}$.*
4. *The element $w_{\mathfrak{a}^+}$ belongs to H^o.*
5. *For every $j = 1, \ldots, d$ one has $\mathrm{sg}_o(\ell_j) = \mathrm{sg}_o(\ell_{d-j+1})$.*

Proof. The implication *1.⇒2.* is straightforward. Now if $\iota_{\mathfrak{b}^+}(\mathfrak{a}^+) = \mathfrak{a}^+$ holds one has

$$\mathfrak{a}^+ = w_{\mathfrak{b}^+} \cdot (-\mathfrak{a}^+) = w_{\mathfrak{b}^+} w_{\mathfrak{a}^+} \cdot \mathfrak{a}^+$$

and therefore $w_{\mathfrak{b}^+} = w_{\mathfrak{a}^+}$. Since $w_{\mathfrak{b}^+}$ belongs to H^o the implication *3.⇒4.* follows. Also, the implication *4.⇒5.* is direct. Finally, assume that

$$\mathrm{sg}_o(\ell_j) = \mathrm{sg}_o(\ell_{d-j+1}) = \mathrm{sg}_o(w_{\mathfrak{a}^+} \cdot \ell_j)$$

holds for every $j = 1, \ldots, d$. By Lemma 6.2.3 we have that $w_{\mathfrak{a}^+}$ belongs to H^o. Then $w_{\mathfrak{b}^+} w_{\mathfrak{a}^+}$ belongs to H^o and takes $\mathfrak{a}^+$ into a $\mathfrak{b}^+$-compatible Weyl chamber. Lemma 6.2.6 implies then $w_{\mathfrak{b}^+} = w_{\mathfrak{a}^+}$ and therefore the equality $\iota_{\mathfrak{b}^+} = \iota_{\mathfrak{a}^+}$.

□

Corollary 6.2.8. *There exists a $\mathfrak{b}^+$-compatible Weyl chamber $\mathfrak{a}^+$ satisfying*

$$\iota_{\mathfrak{b}^+}(\mathfrak{a}^+) = \mathfrak{a}^+$$

if and only if either d is odd, or d is even and p is also even.

□

6.3 Invariant forms on exterior powers

Certain objects of this part of the thesis (notably o-Busemann cocycles and o-Gromov products) will be introduced by appealing to the exterior power representations Λ^j of G. It is then worthwhile to observe that the restriction to $\mathrm{H}^o \cong \mathrm{PSO}(p,q)$ of these representations still preserves a non degenerate quadratic form. Indeed, for each $j = 1, \ldots, d$ define the bilinear form $\langle \cdot, \cdot \rangle_{o_j}$ on the exterior power $\Lambda^j V$ of V by the formula

$$\langle v_1 \wedge \cdots \wedge v_j, v_1' \wedge \cdots \wedge v_j' \rangle_{o_j} := \prod_{i=1}^{j} \langle v_i, v_i' \rangle_o.$$

This form is invariant under the action of $\Lambda^j \mathrm{H}^o$ and if $\mathcal{B} = \{v_1, \ldots, v_d\}$ is an $\langle \cdot, \cdot \rangle_o$-orthonormal basis of V, then the basis

$$\Lambda^j \mathcal{B} := \{v_{i_1} \wedge \cdots \wedge v_{i_j}\}_{1 \leq i_1 < \cdots < i_j \leq d}$$

of $\Lambda^j V$ is $\langle \cdot, \cdot \rangle_{o_j}$-orthonormal. In particular, $\langle \cdot, \cdot \rangle_{o_j}$ is non degenerate.

We denote by o_j the ray in $\mathcal{Q}(\Lambda^j V)$ generated by $\langle \cdot, \cdot \rangle_{o_j}$. Note that for every $g \in \mathrm{G}$ one has

$$^{\star_{o_j}}(\Lambda^j g) = \Lambda^j(^{\star_o} g), \tag{6.3.1}$$

and therefore the following equality holds:

$$\sigma^{o_j}(\Lambda^j g) = \Lambda^j \sigma^o(g). \tag{6.3.2}$$

6.4 Generic flags

The relevant notion of this section, introduced in Subsection 6.4.2, is the notion of *o-generic flags*: they provide a concrete way of working with the open orbits of the action of H^o on $\mathsf{F}(V)$. This notion is built from the existence of an involution

$$\cdot^{\perp_o} : \mathsf{F}(V) \to \mathsf{F}(V),$$

which is introduced in Subsection 6.4.1. In Subsection 6.4.4 we provide a link between o-generic flags and $\mathfrak{b}^+$-compatible Weyl chambers. This link will be of central importance in the next chapter, notably in Subsection 7.2.2.

6.4.1 Involution on $\mathsf{F}(V)$

The following lemma is direct.

Lemma 6.4.1. *Let $j = 1, \ldots, d$. For every g in G and every j-dimensional subspace π in $\mathsf{Gr}_j(V)$ one has*

$$\sigma^o(g) \cdot (\pi^{\perp_o}) = (g \cdot \pi)^{\perp_o} = g \cdot (\pi^{\perp_{g^{-1} \cdot o}}).$$

In particular, the following holds:

$$(g \cdot \pi)^{\perp_{g \cdot o}} = g \cdot (\pi^{\perp_o}).$$

□

Given a full flag $\xi = (\xi^1, \ldots, \xi^d)$ in $\mathsf{F}(V)$ we denote by $\xi^{\perp_o}$ the complete flag of V defined by the equalities

$$(\xi^{\perp_o})^j := (\xi^{d-j})^{\perp_o}$$

for $j = 1, \ldots, d$.

Corollary 6.4.2. *For every g in G and every ξ in $\mathsf{F}(V)$ one has*

$$\sigma^o(g) \cdot (\xi^{\perp_o}) = (g \cdot \xi)^{\perp_o} = g \cdot (\xi^{\perp_{g^{-1} \cdot o}}).$$

In particular, the following holds:

$$(g \cdot \xi)^{\perp_{g \cdot o}} = g \cdot (\xi^{\perp_o}).$$

□

Action on (Cartan) attractors

As we show in the next chapter, the subset $\mathscr{B}_{o,\mathrm{G}}$ coincides with the set of elements $g \in \mathrm{G}$ for which $\sigma^o(g^{-1})g$ is proximal on $\mathsf{F}(V)$. We now compute the Cartan attractor (resp. attractive fixed point in $\mathsf{F}(V)$) of $\sigma^o(g^{-1})$ in terms of the Cartan repellor (resp. repelling fixed point in $\mathsf{F}(V)$) of g, provided g has full gaps (resp. is proximal on $\mathsf{F}(V)$). This will be an important ingredient for understanding the dynamics of elements of the form

$$\sigma^o(g^{-1})g$$

acting on $\mathsf{F}(V)$.

Let $\mathfrak{a}^+ \subset \mathfrak{b}$ be a closed Weyl chamber of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$. Let a^τ (resp. λ) be the Cartan (resp. Jordan) projection of G. Since σ^o preserves K^τ and acts as $-\mathrm{id}$ on $\mathfrak{b}$, Lemma 6.4.1 implies the following.

Corollary 6.4.3.

1. *The following equality holds:*

$$a^\tau \circ \sigma^o = \iota_{\mathfrak{a}^+} \circ a^\tau.$$

In particular, if $g \in \mathrm{G}$ has a gap of index Δ then $\sigma^o(g^{-1})$ also has a gap of index Δ and one has

$$U^\tau(\sigma^o(g^{-1})) = S^\tau(g)^{\perp_o}.$$

2. *The following equality holds:*

$$\lambda \circ \sigma^o = \iota_{\mathfrak{a}^+} \circ \lambda.$$

In particular, if g in G is proximal on $\mathsf{F}(V)$ then $\sigma^o(g^{-1})$ is also proximal on $\mathsf{F}(V)$ and one has

$$\sigma^o(g^{-1})_+ = (g_-)^{\perp_o}.$$

□

6.4.2 Genericity of flags with respect to the basepoint o

A full flag $\xi \in \mathsf{F}(V)$ is said to be *o-generic* if it is transverse to $\xi^{\perp_o}$. Equivalently, we have the following.

Lemma 6.4.4. *Let $\xi = (\xi^1, \ldots, \xi^d)$ be an element of $\mathsf{F}(V)$. Then the following are equivalent:*

1. *The flag ξ is o-generic.*
2. *For every $j = 1, \ldots, d$, the restriction of the form o to ξ^j is non degenerate.*
3. *There exists a unique o-orthogonal ordered basis of lines*

$$\{\ell_1^o(\xi), \ldots, \ell_d^o(\xi)\}$$

of V such that the equality

$$\xi^j = \ell_1^o(\xi) \oplus \cdots \oplus \ell_j^o(\xi)$$

holds for every $j = 1, \ldots, d$.

4. *For every $j = 1, \ldots, d$, the line $\Lambda^j \xi$ is transverse to the hyperplane $(\Lambda^j \xi)^{\perp_{o_j}}$.*

Proof. Suppose that ξ is o-generic and let $j = 1, \ldots, d$. By definition ξ^j is transverse to $(\xi^{\perp_o})^{d-j} = (\xi^j)^{\perp_o}$ and thus the restriction of o to ξ^j is non degenerate.

Now suppose the *2.* holds. Define $\ell_1^o(\xi) := \xi^1$ and suppose by induction that we have $\ell_1^o(\xi), \ldots, \ell_j^o(\xi)$ as in the statement, for some $j < d$. Define $\ell_{j+1}^o(\xi)$ to be the orthogonal complement, with respect to the form o, of ξ^j inside ξ^{j+1}. Since the restriction of o to ξ^j is non degenerate, we have that $\ell_{j+1}^o(\xi)$ is transverse to ξ^j and this shows existence. Uniqueness is straightforward.

Assume that *3.* is satisfied. Since

$$\{\ell_{i_1}^o(\xi) \wedge \cdots \wedge \ell_{i_j}^o(\xi)\}_{1 \leq i_1 < \cdots < i_j \leq d}$$

is an o_j-orthogonal basis of lines of $\Lambda^j V$, we conclude that *4.* must hold.

To finish we show the implication *4.⇒1.* By contradiction, suppose that *4.* holds and that there exists $j = 1, \ldots, d-1$ for which the intersection $\xi^j \cap (\xi^j)^{\perp_o}$ is non trivial. Take a (necessarily o-isotropic) line ℓ in $\xi^j \cap (\xi^j)^{\perp_o}$. This implies that the line $\Lambda^j \xi$ is o_j-isotropic and therefore contained in $(\Lambda^j \xi)^{\perp_{o_j}}$.

□

Remark 6.4.5.

- Every pair $(\xi_1, \xi_2) \in \mathsf{F}(V)^{(2)}$ of transverse flags determines an ordered basis of lines $\{\ell_1(\xi_1, \xi_2), \dots, \ell_d(\xi_1, \xi_2)\}$ of V defined by

$$\ell_j(\xi_1, \xi_2) := \xi_1^j \cap \xi_2^{d-j+1}$$

for every $j = 1, \dots, d$. If a flag ξ is o-generic, then the ordered basis of lines given by Lemma 6.4.4 coincides with the one induced by the pair $(\xi, \xi^{\perp_o}) \in \mathsf{F}(V)^{(2)}$.

- Fix $j = 1, \dots, d$. Recall that one has a map

$$\Lambda_*^j : \mathsf{F}(V) \to \mathsf{P}(\Lambda^j V^*)$$

which is equivariant (Subsection B.2.1). Let ξ be an o-generic flag. Then the following equality holds:

$$\Lambda_*^j(\xi^{\perp_o}) = (\Lambda^j \xi)^{\perp_{o_j}}.$$

In particular, if $\xi' \in \mathsf{F}(V)$ is a flag transverse to ξ then the hyperplane

$$(\Lambda^j(\xi^{\perp_o}))^{\perp_{o_j}}$$

is transverse to the line $\Lambda^j \xi'$.

- Let g be an element of G. A flag ξ is o-generic if and only if $g \cdot \xi$ is $(g \cdot o)$-generic (Corollary 6.4.2).

◇

6.4.3 Open orbits of point-stabilizers

Denote by $\mathsf{F}(V)^o$ the subset of $\mathsf{F}(V)$ consisting on o-generic flags.

Proposition 6.4.6. *The set $\mathsf{F}(V)^o$ coincides with the union of open orbits of the action of H^o on $\mathsf{F}(V)$.*

Proof. By Corollary 6.4.2 we know that $\mathsf{F}(V)^o$ is a union of H^o-orbits.

Further, for every $j = 1, \dots, d$, the map that asigns to each j-uple of linearly independent vectors $(v_1, \dots, v_j) \in V^j$ the determinant

$$\det\left(\langle v_i, v_k \rangle_o\right)_{i,k=1,\dots,j}$$

is analytic and therefore the preimage of 0 by this map has empty interior. By Lemma 6.4.4 we conclude that $\mathsf{F}(V)^o$ is open and, since the map

$$(v_1, \ldots, v_j) \mapsto \mathrm{span}\{v_1, \ldots, v_j\} \in \mathsf{Gr}_j(V)$$

is open, the set $\mathsf{F}(V) \setminus \mathsf{F}(V)^o$ must have empty interior.

□

The proof of the following proposition is direct.

Proposition 6.4.7. *Let ξ and ξ' be two o-generic flags. Then the following are equivalent:*

1. *The flags ξ and ξ' belong to the same orbit of the action $\mathrm{H}^o \curvearrowright \mathsf{F}(V)$.*

2. *For every $j = 1, \ldots, d$, the signature of o restricted to ξ^j coincides with the signature of o restricted to ξ'^j.*

3. *For every $j = 1, \ldots, d$, one has $\mathrm{sg}_o(\ell_j^o(\xi)) = \mathrm{sg}_o(\ell_j^o(\xi'))$.*

4. *For every $j = 1, \ldots, d$, one has*

$$\mathrm{sg}_{o_j}(\Lambda^j \xi) = \mathrm{sg}_{o_j}(\Lambda^j \xi').$$

□

6.4.4 Open orbits and compatible Weyl chambers

We now describe the link between open orbits of the action

$$\mathrm{H}^o \curvearrowright \mathsf{F}(V)$$

and the set of $\mathfrak{b}^+$-compatible Weyl chambers. In order to do that, recall that there exists a W-equivariant identification between the set of Weyl chambers of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$ and the set of (o-generic) flags determined by the lines of $\mathcal{C}$. If $\mathfrak{a}^+ \subset \mathfrak{b}$ is such a Weyl chamber the corresponding flag is denoted by $\xi_{\mathfrak{a}^+}$. Conversely, the Weyl chamber of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$ determined by a flag ξ spanned by $\mathcal{C}$ will be denoted by $\mathfrak{a}_\xi^+$.

A flag $\xi \in \mathsf{F}(V)$ is said to be $\mathfrak{b}^+$-*compatible* if it is spanned by the elements of $\mathcal{C}$ and the Weyl chamber $\mathfrak{a}_\xi^+$ is $\mathfrak{b}^+$-compatible.

Matsuki [43, Section 3] and Rossmann [59, Theorem 13 and Corollaries 15 to 17] study the structure of the quotient space

$$\mathrm{P}\backslash\mathsf{X}$$

where X is any symmetric space of a semisimple Lie group and P is a parabolic subgroup. They give a Lie theoretic description of this set and in particular provide a description of the subset consisting on open orbits. In our concrete case, we have the following.

Proposition 6.4.8. *The map*

$$\xi \mapsto \mathsf{F}(V)^o_\xi := \mathrm{H}^o \cdot \xi$$

defines a one to one correspondence between the set of $\mathfrak{b}^+$-compatible flags and the set consisting on open orbits of the action $\mathrm{H}^o \curvearrowright \mathsf{F}(V)$.

Let $\mathfrak{a}^+$ be a $\mathfrak{b}^+$-compatible Weyl chamber. We will often use the notation

$$\mathsf{F}(V)^o_{\mathfrak{a}^+} := \mathsf{F}(V)^o_{\xi_{\mathfrak{a}^+}}.$$

Proof of Proposition 6.4.8. We first show injectivity. Let ξ and ξ' be two $\mathfrak{b}^+$-compatible flags in the same H^o-orbit. By Proposition 6.4.7 we have

$$\mathrm{sg}_o(\ell^o_j(\xi)) = \mathrm{sg}_o(\ell^o_j(\xi'))$$

for every $j = 1, \dots, d$. Now note that, us unordered sets, one has the equalities:

$$\{\ell^o_1(\xi), \dots, \ell^o_d(\xi)\} = \mathcal{C} = \{\ell^o_1(\xi'), \dots, \ell^o_d(\xi')\}.$$

Therefore Lemma 6.2.3 implies the existence of an element $\hat{w} \in \hat{\mathrm{W}} \cap \mathrm{H}^o$ such that

$$w \cdot \ell^o_j(\xi) = \ell^o_j(\xi')$$

holds for every $j = 1, \dots, d$, that is, $w \cdot \mathfrak{a}^+_\xi = \mathfrak{a}^+_{\xi'}$. Thanks to Lemma 6.2.6 we know that $w = 1$ and therefore $\xi = \xi'$.

For surjectivity, let ξ' be any o-generic flag. By Lemma 6.4.4 we can find an element $h \in \mathrm{H}^o$ such that

$$h \cdot \ell^o_j(\xi') \in \mathcal{C}$$

for every $j = 1, \dots, d$. That is, $h \cdot \xi'$ determines a Weyl chamber of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$. By Lemma 6.2.6 the proof is complete.

□

We finish this chapter by giving a necessary and sufficient condition for the existence of a $\mathfrak{b}^+$-compatible Weyl chamber which is fixed by the opposition involution $\iota_{\mathfrak{b}^+}$ (this condition can then be added to Corollary 6.2.7). Given a $\mathfrak{b}^+$-compatible flag ξ we define $\iota_{\mathfrak{b}^+}(\xi)$ to be the ($\mathfrak{b}^+$-compatible) flag determined by the Weyl chamber

$$\iota_{\mathfrak{b}^+}(\mathfrak{a}^+_\xi) = -w_{\mathfrak{b}^+} \cdot \mathfrak{a}^+_\xi.$$

By Corollary 6.4.2 the map

$$\cdot^{\perp_o} : \mathsf{F}(V) \to \mathsf{F}(V)$$

takes H^o-orbits into H^o-orbits and since

$$\iota_{\mathfrak{b}^+}(\xi) = w_{\mathfrak{b}^+} \cdot (\xi^{\perp_o}) \in (\mathrm{H}^o \cdot \xi)^{\perp_o},$$

we conclude that the following equality holds:

$$(\mathsf{F}(V)^o_\xi)^{\perp_o} = \mathsf{F}(V)^o_{\iota_{\mathfrak{b}^+}(\xi)}.$$

Corollary 6.4.9. *Let $\mathfrak{a}^+$ be a $\mathfrak{b}^+$-compatible Weyl chamber. Then the following are equivalent:*

1. *One has the equality $(\mathsf{F}(V)^o_{\mathfrak{a}^+})^{\perp_o} = \mathsf{F}(V)^o_{\mathfrak{a}^+}$.*
2. *One has the equality $\iota_{\mathfrak{b}^+}(\mathfrak{a}^+) = \mathfrak{a}^+$.*

□

Chapter 7

Generalized Cartan decomposition

Fix a point $o \in \mathsf{Q}_{p,q}$ and let $\mathscr{B}_o$ be the space of o-orthogonal bases of lines of V. The first goal of this chapter is to show that the set

$$\mathscr{B}_{o,\mathrm{G}} := \{g \in \mathrm{G} : \quad \exists\, \mathcal{C} \in \mathscr{B}_o \text{ such that } g \cdot \mathcal{C} \in \mathscr{B}_o\}$$

admits a decomposition analogue to the Cartan decomposition of G, and to define and associated projection b^o. We then begin the study of this decomposition, discussing geometric interpretations and studying it for elements $g \in \mathrm{G}$ having full gaps and o-generic Cartan attractor and repellor (Section 7.2). The contents of Subsection 7.3.3 will be crucial for our understanding the new decomposition: we will be able to interpret the vector $b^o(g)$ using the Jordan projection of $\lambda(\sigma^o(g^{-1})g)$ and to estimate $b^o(g)$ in terms of the Cartan projection $a^\tau(g)$.

7.1 Cartan decomposition for $\mathscr{B}_{o,\mathrm{G}}$

Fix a point τ in S^o, a maximal subalgebra $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ and a Weyl chamber $\mathfrak{b}^+ \subset \mathfrak{b}$ of the system $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$. Let $\mathcal{C} \in \mathscr{B}_o$ be the element determined by the choice of $\mathfrak{b}$.

7.1.1 Statement of the result

Proposition 7.1.1. *Let g be an element of* G*. Then the following are equivalent:*

1. *The element g belongs to $\mathscr{B}_{o,\mathrm{G}}$.*
2. *The element g belongs to $\mathrm{H}^o\hat{\mathrm{W}}\exp(\mathfrak{b}^+)\mathrm{H}^o$.*

3. The element $\sigma^o(g^{-1})g = {}^{\star_o}gg$ is diagonalizable[1].

In this case, let $\tilde{\mathcal{C}}$ be an element of $\mathscr{B}_o$. Then $g \cdot \tilde{\mathcal{C}}$ belongs to $\mathscr{B}_o$ if and only if $\tilde{\mathcal{C}}$ diagonalizes $\sigma^o(g^{-1})g$.

Note that the element $\sigma^o(g^{-1})g$ is fixed by the o-adjoint operator or, in short, it is $\star_o$-*symmetric*. Since the form o is not definite, $\star_o$-symmetric are not necessarily diagonalizable. However, when they are then they are diagonalizable in an o-orthogonal basis.

A decomposition $g = h\hat{w}\exp(X)\tilde{h}$ of an element $g \in \mathscr{B}_{o,\mathrm{G}}$, where $h, \tilde{h} \in \mathrm{H}^o$, $\hat{w} \in \hat{\mathrm{W}}$ and $X \in \mathfrak{b}^+$ will be called a (p,q)-*Cartan decomposition* of g.

Proof of Proposition 7.1.1. Suppose first that g belongs to $\mathscr{B}_{o,\mathrm{G}}$ and let $\tilde{\mathcal{C}} \in \mathscr{B}_o$ be an element such that $g \cdot \tilde{\mathcal{C}} \in \mathscr{B}_o$. Write $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}^+ \sqcup \tilde{\mathcal{C}}^-$ the decomposition of $\tilde{\mathcal{C}}$ into positive and negative lines for the form o. There exists $h \in \mathrm{H}^o$ such that $h^{-1}g \cdot \tilde{\mathcal{C}} = \mathcal{C}$. Further, we can take $\hat{w}$ in $\hat{\mathrm{W}}$ such that $\hat{w}^{-1}h^{-1}g \cdot \tilde{\mathcal{C}}^{\pm} = \mathcal{C}^{\pm}$. Now let $\tilde{h} \in \mathrm{H}^o$ such that $\tilde{h}^{-1} \cdot \mathcal{C} = \tilde{\mathcal{C}}$. We can further assume that $\hat{w}^{-1}h^{-1}g\tilde{h}^{-1}$ fixes each line of $\mathcal{C}$. Then one has

$$\hat{w}^{-1}h^{-1}g\tilde{h}^{-1} = m\exp(X)$$

for some $X \in \mathfrak{b}$ and $m \in \mathrm{M}$. Since M is contained in $\hat{\mathrm{W}}$ and X is conjugate to an element in $\mathfrak{b}^+$, by an element in $\hat{\mathrm{W}} \cap \mathrm{H}^o$, we conclude that g belongs to $\mathrm{H}^o\hat{\mathrm{W}}\exp(\mathfrak{b}^+)\mathrm{H}^o$.

Now suppose that g admits a (p,q)-Cartan decomposition

$$g = h\hat{w}\exp(X)\tilde{h}.$$

Then

$$\sigma^o(g^{-1})g = \tilde{h}^{-1}\exp(X)\sigma^o(w^{-1})w\exp(X)\tilde{h} \qquad (7.1.1)$$

and now it suffices to show that $\exp(X)\sigma^o(w^{-1})w\exp(X)$ fixes the elements of $\mathcal{C}$. But this follows from the definition of $\mathcal{C}$ and Lemma 6.2.5.

Finally, suppose that $\sigma^o(g^{-1})g$ is diagonalizable. Then it is diagonalizable in an o-orthogonal basis of lines $\tilde{\mathcal{C}}$. Given $\ell \neq \ell'$ in $\tilde{\mathcal{C}}$ we have

$$\langle g \cdot \ell, g \cdot \ell' \rangle_o = \langle \sigma^o(g^{-1})g \cdot \ell, \ell' \rangle_o = \langle \ell, \ell' \rangle_o = 0,$$

where the equalities hold up to scalar multiples. Hence $g \cdot \tilde{\mathcal{C}} \in \mathscr{B}_o$.

□

[1] Formally, the element $\sigma^o(g^{-1})g$ is not a linear transformation of V but rather a projective class of linear transformations. Nevertheless it makes sense to speak about "diagonalizable" elements of G (c.f. Subsection B.2.2).

7.1.2 (p,q)-Cartan projection

Proposition 7.1.2. *Let g be an element of $\mathscr{B}_{o,\mathrm{G}}$ and write*

$$h_1\hat{w}_1\exp(X_1)\tilde{h}_1 = g = h_2\hat{w}_2\exp(X_2)\tilde{h}_2$$

two (p,q)-Cartan decompositions of g. Then $X_1 = X_2$.

Proof. Write

$$V = \bigoplus_{\mu} V^o_\mu(g),$$

where $V^o_\mu(g)$ is the eigenspace of $\sigma^o(g^{-1})g$ associated to the eigenvalue μ. It can be seen that the restriction of o to $V^o_\mu(g)$ is non degenerate, we let (p_μ, q_μ) be the signature of this restriction.

For $i = 1,2$ let $m_i := \sigma^o(\hat{w}_i^{-1})\hat{w}_i \in \mathrm{M}$. Since $\sigma^o(g^{-1})g = \tilde{h}_i^{-1}m_i\exp(X_i)^2\tilde{h}_i$, the set of eigenvalues of $\exp(X_i)^2$ coincides with

$$\{|\mu| : \quad V^o_\mu(g) \neq 0\}.$$

Moreover we conclude that $V^o_\mu(g) = \tilde{h}_i^{-1}\cdot V^i_\mu$, where V^i_μ is the eigenspace of $m_i\exp(X_i)^2$ associated to the eigenvalue μ.

Write $V^i_{|\mu|} := V^i_\mu \oplus V^i_{-\mu}$, the eigenspace of $\exp(X_i)^2$ associated to the eigenvalue $|\mu|$. We claim that $V^i_{|\mu|}$ does not depend on $i = 1,2$. Indeed, let μ be an eigenvalue of $\sigma^o(g^{-1})g$ such that $|\mu|$ is maximal. Then $V^i_{|\mu|}$ coincides with the subspace of V spanned by the first[2] $p_\mu + p_{-\mu}$ lines of $\mathcal{C}^+$ and the first $q_\mu + q_{-\mu}$ lines of $\mathcal{C}^-$. This description does not depend on $i = 1,2$ and the claim follows by an inductive argument.

Set $V_{|\mu|} := V^1_{|\mu|} = V^2_{|\mu|}$. For every μ and every $i = 1,2$ the restriction $\exp(X_i)^2|_{V_{|\mu|}}$ equals $|\mu|\mathrm{id}_{V_{|\mu|}}$ and this completes the proof.

□

Remark 7.1.3. Even though is not needed, let us remark that one can prove the following. Write

$$h_1\hat{w}_1\exp(X)\tilde{h}_1 = g = h_2\hat{w}_2\exp(X)\tilde{h}_2$$

two (p,q)-Cartan decompositions of an element $g \in \mathscr{B}_{o,\mathrm{G}}$. Then there exists an element $\hat{w} \in \hat{\mathrm{W}} \cap \mathrm{H}^o$ such that $[w_1] = [w_2 w]$ in $[\mathrm{W}]$. If moreover X belongs to $\mathrm{int}(\mathfrak{b}^+)$, then $[w_1] = [w_2]$ in $[\mathrm{W}]$. In this case, there exist m and m' in M and $\hat{w} \in \hat{\mathrm{W}} \cap \mathrm{H}^o$ such that

$$h_1\hat{w}m' = h_2, \qquad \tilde{h}_1 = m^{-1}\hat{w}_2^{-1}m'\hat{w}_2\tilde{h}_2.$$

[2] Recall that the choice of $\mathfrak{b}^+$ induces a total order on $\mathcal{C}^\pm$.

◇

Define the (p,q)-*Cartan projection*

$$b^o : \mathscr{B}_{o,\mathrm{G}} \to \mathfrak{b}^+,$$

where $g = h\hat{w}\exp(b^o(g))\tilde{h}$ is a (p,q)-Cartan decomposition of $g \in \mathscr{B}_{o,\mathrm{G}}$.

7.2 (p,q)-Cartan decomposition for elements with gaps

We now begin a more detailed study of the dynamics of matrices of the form $\sigma^o(g^{-1})g$ under the assumption that the element $g \in \mathrm{G}$ has a gap of index Δ and that the flags $U^\tau(g)$ and $S^\tau(g)$ are o-generic. We start by introducing an analogue of these flags in our context and establishing some initial properties for them (Subsection 7.2.1). Then we compute the $\mathfrak{b}^+$-compatible Weyl chamber that contains $b^o(g)$ and the $\hat{\mathrm{W}}$-coordinate of g in a (p,q)-Cartan decomposition (Subsection 7.2.2).

7.2.1 (p,q)-Cartan attractors

Let $g \in \mathscr{B}_{o,\mathrm{G}}$ be an element such that $\sigma^o(g^{-1})g$ is loxodromic. It follows from Lemma 6.2.5 and equation (7.1.1) that this is equivalent to the existence of a $\mathfrak{b}^+$-compatible Weyl chamber $\mathfrak{a}^+$ for which one has

$$g \in \mathrm{H}^o\hat{\mathrm{W}}\exp(\mathrm{int}(\mathfrak{a}^+))\mathrm{H}^o,$$

where $\mathrm{int}(\mathfrak{a}^+)$ denotes the interior of $\mathfrak{a}^+$. In this case we set

$$U^o(g) := (g\sigma^o(g^{-1}))_+ \in \mathsf{F}(V).$$

By similar reasons the element $\sigma^o(g)g^{-1}$ is loxodromic as well and we denote

$$S^o(g) := U^o(g^{-1}).$$

Note that

$$S^o(g) = (\sigma^o(g^{-1})g)_-.$$

Remark 7.2.1. Suppose that $g = h\hat{w}\exp(X)\tilde{h}$ is a (p,q)-Cartan decomposition of g such that $X \in \mathrm{int}(\mathfrak{a}^+)$ for some Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{b}^+$. Then

$$U^o(g) = h\hat{w}\cdot\xi_{\mathfrak{a}^+} \text{ and } S^o(g) = \tilde{h}^{-1}\cdot\xi_{-\mathfrak{a}^+} = \tilde{h}^{-1}\cdot(\xi_{\mathfrak{a}^+}^{\perp_o}).$$

In particular, both $U^o(g)$ and $S^o(g)$ belong to the set $\mathsf{F}(V)^o$ of o-generic flags on V.

◇

Recall that the choice of τ induces a distance in all exterior powers $\Lambda^j V$ and in the flag manifold $\mathsf{F}(V)$. Recall also that an (r,ε)-loxodromic element is an element of G that is (r,ε)-proximal on $\mathsf{F}(V)$ (Subsection 1.5.2). We have the following analogue of Lemma 4.3.2 in this setting.

Lemma 7.2.2. *Let $C \subset \mathsf{F}(V)^o$ be a compact set. Then there exist $0 < \varepsilon_0 \leq r_0$ such that for every $0 < \varepsilon \leq r$ with $\varepsilon \leq \varepsilon_0$ and $r \leq r_0$ there exists a positive L with the following property: fix a $\mathfrak{b}^+$-compatible Weyl chamber $\mathfrak{a}^+$. For every $g \in \mathrm{G}$ for which*

$$\min_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} \alpha(a^\tau(g)) > L$$

holds, and such that $U^\tau(g) \in C$ and $S^\tau(g) \in C$, then one has that $\sigma^o(g^{-1})g$ is $(2r, 2\varepsilon)$-loxodromic. Further, given a positive δ the number ε_0 can be chosen in such a way that

$$d(U^o(g), U^\tau(g)) < \delta \text{ and } d(S^o(g), S^\tau(g)) < \delta.$$

Proof. The proof is essentially the same as that of Lemma 4.3.2. Namely, because of Lemma 6.4.4 there exists a positive r_0 for which for every $j = 1, \ldots, d-1$ and every $\xi \in C$ one has

$$d(\Lambda^j \xi, (\Lambda^j \xi)^{\perp_{o_j}}) \geq 6r_0 \text{ and } d(\Lambda^j(\xi^{\perp_o}), (\Lambda^j(\xi^{\perp_o}))^{\perp_{o_j}}) \geq 6r_0.$$

By Remark 6.4.5 this implies that

$$d(\Lambda^j \xi, \Lambda^j_*(\xi^{\perp_o})) \geq 6r_0 \text{ and } d(\Lambda^j(\xi^{\perp_o}), \Lambda^j_* \xi) \geq 6r_0 \tag{7.2.1}$$

holds for every $\xi \in C$. We now find an $\varepsilon_0 \leq r_0$ for which for every $j = 1, \ldots, d-1$ and every $\xi \in C$ one has

$$b_{\varepsilon_0}(\Lambda^j \xi) \subset B_{\varepsilon_0}(\Lambda^j_*(\xi^{\perp_o})) \text{ and } b_{\varepsilon_0}(\Lambda^j(\xi^{\perp_o})) \subset B_{\varepsilon_0}(\Lambda^j_* \xi). \tag{7.2.2}$$

Fix $0 < \varepsilon \leq r$ with $\varepsilon \leq \varepsilon_0$ and $r \leq r_0$. We can find a positive L for which for every g such that

$$\min_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} \alpha(a^\tau(g)) > L$$

one has

$$\Lambda^j g \cdot B_\varepsilon\left(\Lambda^j_* S^\tau(g)\right) \subset b_\varepsilon(\Lambda^j U^\tau(g)). \tag{7.2.3}$$

for every $j = 1, \ldots, d-1$. Further, by Corollary 6.4.3 we have as well

$$\Lambda^j \sigma^o(g^{-1}) \cdot B_\varepsilon\left(\Lambda^j_* S^\tau(\sigma^o(g^{-1}))\right) \subset b_\varepsilon(\Lambda^j U^\tau(\sigma^o(g^{-1}))). \tag{7.2.4}$$

Now suppose moreover that $U^\tau(g) \in C$ and $S^\tau(g) \in C$. By (7.2.1) and Corollary 6.4.3 we have

$$d\left(\Lambda^j U^\tau(\sigma^o(g^{-1})), \Lambda^j_* S^\tau(g)\right) \geq 6r. \tag{7.2.5}$$

Now by (7.2.3) we have

$$\Lambda^j(\sigma^o(g^{-1})g) \cdot B_\varepsilon(\Lambda^j_* S^\tau(g)) \subset \Lambda^j \sigma^o(g^{-1}) \cdot b_\varepsilon(\Lambda^j U^\tau(g))$$

and by Corollary 6.4.3 and (7.2.2) is contained in

$$\Lambda^j \sigma^o(g^{-1}) \cdot B_\varepsilon(\Lambda^j_* S^\tau(\sigma^o(g^{-1}))).$$

By (7.2.4) we have therefore

$$\Lambda^j(\sigma^o(g^{-1})g) \cdot B_\varepsilon(\Lambda^j_* S^\tau(g)) \subset b_\varepsilon(\Lambda^j U^\tau(\sigma^o(g^{-1}))).$$

This inclusion together with (7.2.5) and Benoist's Lemma B.2.3 gives that $\sigma^o(g^{-1})g$ is $(2r, 2\varepsilon)$-loxodromic and

$$d((\sigma^o(g^{-1})g)_-, S^\tau(g)) \leq \varepsilon.$$

Therefore the distance $d(S^o(g), S^\tau(g))$ can be made arbitrarily close to zero.

Finally, working with $g\sigma^o(g^{-1})$ (instead of $\sigma^o(g^{-1})g$) the estimate

$$d(U^o(g), U^\tau(g)) \leq \varepsilon$$

also follows.

□

7.2.2 Computation of the Weyl chamber and the $\hat{\mathrm{W}}$-coordinate

Let ξ and $\tilde{\xi}$ be two flags spanned by $\mathcal{C}$. We denote by $w_{\xi\tilde{\xi}}$ the unique element of the Weyl group W for which one has

$$w_{\xi\tilde{\xi}} \cdot \tilde{\xi} = \xi.$$

Recall that ξ is said to be $\mathfrak{b}^+$-compatible if the associated Weyl chamber $\mathfrak{a}^+_\xi$ is $\mathfrak{b}^+$-compatible.

Proposition 7.2.3. *Let ξ_s and ξ_u be two $\mathfrak{b}^+$-compatible flags. Fix two compact sets $C_s \subset \mathsf{F}(V)^o_{\xi_s}$ and $C_u \subset \mathsf{F}(V)^o_{\xi_u}$. Then there exists a positive L with the following property: let $\mathfrak{a}^+$ be a $\mathfrak{b}^+$-compatible Weyl chamber. For every $g \in \mathrm{G}$ for which*

$$\min_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} \alpha(a^\tau(g)) > L$$

holds, and such that $S^\tau(g) \in C_s$ and $U^\tau(g) \in C_u$, one has

$$g \in \mathrm{H}^o w_{\xi_u \iota_{\mathfrak{b}^+}(\xi_s)} \exp(\mathrm{int}(\iota_{\mathfrak{b}^+}(\mathfrak{a}^+_{\xi_s})))\mathrm{H}^o.$$

Proof. By Lemma 7.2.2 we can take a positive L such that for every g as in the statement one has

$$U^o(g) \in \mathsf{F}(V)^o_{\xi_u} = \mathrm{H}^o \cdot \xi_u$$

and

$$S^o(g) \in \mathsf{F}(V)^o_{\xi_s} = \mathrm{H}^o \cdot \xi_s.$$

If $h\hat{w}\exp(b^o(g))\tilde{h}$ is a (p,q)-Cartan decomposition of g we then have

$$U^o(g) = h\hat{w} \cdot b^o(g)_+ \in \mathrm{H}^o \cdot \xi_u$$

and

$$S^o(g) = \tilde{h}^{-1} \cdot b^o(g)_- \in \mathrm{H}^o \cdot \xi_s.$$

In particular,

$$\hat{w} \cdot b^o(g)_+ \in \mathrm{H}^o \cdot \xi_u \text{ and } b^o(g)_- \in \mathrm{H}^o \cdot \xi_s.$$

We conclude that

$$b^o(g)_+ = b^o(g)_-^{\perp_o} \in \mathrm{H}^o \cdot (\xi_s)^{\perp_o} = \mathrm{H}^o \cdot \iota_{\mathfrak{b}^+}(\xi_s),$$

because $w_{\mathfrak{b}^+}$ belongs to H^o. Since both $b^o(g)_+$ and $\iota_{\mathfrak{b}^+}(\xi_s)$ are $\mathfrak{b}^+$-compatible, Proposition 6.4.8 implies

$$b^o(g)_+ = \iota_{\mathfrak{b}^+}(\xi_s)$$

and therefore $b^o(g)$ belongs to $\mathrm{int}(\iota_{\mathfrak{b}^+}(\mathfrak{a}^+_{\xi_s}))$. Further, we have

$$\hat{w} \cdot \iota_{\mathfrak{b}^+}(\xi_s) \in \mathrm{H}^o \cdot \xi_u$$

and Lemma 6.2.3 implies the existence of an element $\hat{w}' \in \hat{\mathrm{W}} \cap \mathrm{H}^o$ such that

$$\hat{w}'\hat{w} \cdot \iota_{\mathfrak{b}^+}(\xi_s) = \xi_u.$$

Then $w'w = w_{\xi_u \iota_{\mathfrak{b}^+}(\xi_s)}$ and the proof is complete.

□

Corollary 7.2.4. *Let ξ be a $\mathfrak{b}^+$-compatible flag and $C \subset \mathsf{F}(V)^o_\xi$ a compact set. Then there exists a positive L with the following property: let $\mathfrak{a}^+$ be a $\mathfrak{b}^+$-compatible Weyl chamber. For every $g \in \mathrm{G}$ for which*

$$\min_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} \alpha(a^\tau(g)) > L$$

holds, and such that $S^\tau(g) \in C$ and $U^\tau(g) \in C$, one has

$$g \in \mathrm{H}^o w_{\xi_{\iota_{\mathfrak{b}^+}(\xi)}} \exp(\mathrm{int}(\iota_{\mathfrak{b}^+}(\mathfrak{a}_\xi^+)))\mathrm{H}^o.$$

In particular, if $(\mathsf{F}(V)_\xi^o)^{\perp_o} = \mathsf{F}(V)_\xi^o$ *we have*

$$g \in \mathrm{H}^o \exp(\mathrm{int}(\mathfrak{a}_\xi^+))\mathrm{H}^o.$$

□

Let us emphasize here the fact that, depending on d, p and q, an open orbit $\mathsf{F}(V)_\xi^o$ for which the equality

$$(\mathsf{F}(V)_\xi^o)^{\perp_o} = \mathsf{F}(V)_\xi^o$$

holds may or may not exist (c.f. Corollaries 6.2.8 and 6.4.9).

7.3 Geometric and linear algebraic interpretations

7.3.1 Geometric interpretation in X_G

Note that an element g belongs to $\mathscr{B}_{o,\mathrm{G}}$ if and only if there exists a basis of lines $\mathcal{C}'$ of V which is both o-orthogonal and $(g \cdot o)$-orthogonal. By Proposition 6.1.1, this is equivalent to the existence of a flat of X_G orthogonal both to S^o and $\mathsf{S}^{g \cdot o} = g \cdot \mathsf{S}^o$.

Lemma 7.3.1. *Let g be an element in $\mathscr{B}_{o,\mathrm{G}}$ and $g = h\hat{w}\exp(b^o(g))\tilde{h}$ be a (p,q)-Cartan decomposition of g. Then*

$$h\hat{w}\exp(\mathfrak{b}) \cdot \tau = h\exp(\mathfrak{b}) \cdot \tau$$

is a flat of X_G orthogonal to S^o at $h \cdot \tau$ and to $g \cdot \mathsf{S}^o$ at $h\hat{w}\exp(b^o(g)) \cdot \tau$. In particular, one has

$$d_{\mathsf{X}_\mathrm{G}}(\mathsf{S}^o, g \cdot \mathsf{S}^o) = \|b^o(g)\|_{\mathfrak{b}}.$$

Proof. Indeed, this follows from the fact that $\exp(\mathfrak{b}) \cdot \tau$ is orthogonal to S^o (resp. $\exp(b^o(g)) \cdot \mathsf{S}^o$) at τ (resp. $\exp(b^o(g)) \cdot \tau$) and the fact that $\hat{\mathrm{W}}$ fixes τ and preserves $\exp(\mathfrak{b}) \cdot \tau$. □

7.3.2 Geometric interpretation in $\mathsf{Q}_{p,q}$

Let g be an element in $\mathscr{B}_{o,\mathrm{G}}$ and suppose first that g admits a (p,q)-Cartan decomposition of the form

$$g = h\exp(b^o(g))\tilde{h}.$$

The geometric interpretation of the (p,q)-Cartan projection of g is in this case straightforward: $b^o(g)$ can be interpreted as the "vector" inside the space-like flat

$$h\exp(\mathfrak{b})\cdot o$$

that connects the basepoint o with $g\cdot o$.

Now suppose that g admits a (p,q)-Cartan decomposition of the form

$$g=\hat{w}\exp(b^o(g))\tilde{h}$$

for some $\hat{w}\in\hat{\mathrm{W}}\setminus\mathrm{H}^o$. Then the space-like flat

$$\hat{w}\exp(\mathfrak{b})\cdot o$$

is disjoint from $\exp(\mathfrak{b})\cdot o$ (c.f. Subsection 6.1.2). We can interpret the element $b^o(g)$ in this case as the "vector" connecting the points $\hat{w}\cdot o$ and $g\cdot o$ inside the space-like flat

$$\hat{w}\exp(\mathfrak{b})\cdot o=\exp(\mathfrak{b})\cdot(\hat{w}\cdot o).$$

In the general case $g=h\hat{w}\exp(b^o(g))\tilde{h}$, the vector $b^o(g)$ connects the point $h\hat{w}\cdot o$ with $g\cdot o$ inside the space-like flat $h\hat{w}\exp(\mathfrak{b})\cdot o$.

7.3.3 Linear algebraic interpretation

For the rest of the section we fix a $\mathfrak{b}^+$-compatible Weyl chamber $\mathfrak{a}^+$. Lemma 6.2.5 and equation (7.1.1) imply the following: for every element g of $\mathscr{B}_{o,\mathrm{G}}$ there exists an element $w_g\in\mathrm{W}$ such that

$$b^o(g)=\tfrac{1}{2}w_g\cdot\lambda(\sigma^o(g^{-1})g).$$

In particular, one has:

$$\|b^o(g)\|_{\mathfrak{b}}=\tfrac{1}{2}\|\lambda(\sigma^o(g^{-1})g)\|_{\mathfrak{b}}.$$

Whenever g has a (sufficiently large) gap of index Δ and the Cartan attractor and repellor of g are o-generic, we have the following more precise result.

Proposition 7.3.2. *Let ξ_s be a $\mathfrak{b}^+$-compatible flag and fix compact sets $C_s\subset\mathsf{F}(V)^o_{\xi_s}$ and $C\subset\mathsf{F}(V)^o$. Then there exists a positive L with the following property: for every $g\in\mathrm{G}$ for which*

$$\min_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})}\alpha(a^\tau(g))>L$$

holds, and such that $S^\tau(g) \in C_s$ *and* $U^\tau(g) \in C$*, one has*

$$b^o(g) = \tfrac{1}{2} w_{\iota_{\mathfrak{b}^+}(\xi_s)\xi_{\mathfrak{a}^+}} \cdot \lambda(\sigma^o(g^{-1})g).$$

In particular, if $\iota_{\mathfrak{b}^+}(\xi_s) = \xi_{\mathfrak{a}^+}$ *we have*

$$b^o(g) = \tfrac{1}{2}\lambda(\sigma^o(g^{-1})g).$$

Proof. Apply Proposition 7.2.3 to each intersection of C with each open orbit of the action $\mathrm{H}^o \curvearrowright \mathsf{F}(V)$. For every g as in the statement we know that there exists some $X \in \operatorname{int}(\mathfrak{a}^+)$ such that

$$b^o(g) = w_{\iota_{\mathfrak{b}^+}(\xi_s)\xi_{\mathfrak{a}^+}} \cdot X.$$

Hence

$$\tfrac{1}{2}\lambda(\sigma^o(g^{-1})g) = \lambda(\exp(b^o(g))) = X = (w_{\iota_{\mathfrak{b}^+}(\xi_s)\xi_{\mathfrak{a}^+}})^{-1} \cdot b^o(g).$$

□

Proposition 7.3.3. *Let* ξ_s *be a* $\mathfrak{b}^+$*-compatible flag and fix compact sets* $C_s \subset \mathsf{F}(V)^o_{\xi_s}$ *and* $C \subset \mathsf{F}(V)^o$*. Then there exist positive numbers* L *and* D *with the following property: for every* $g \in \mathrm{G}$ *for which*

$$\min_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} \alpha(a^\tau(g)) > L$$

holds, and such that $S^\tau(g) \in C_s$ *and* $U^\tau(g) \in C$*, one has*

$$\| b^o(g) - w_{\iota_{\mathfrak{b}^+}(\xi_s)\xi_{\mathfrak{a}^+}} \cdot a^\tau(g) \|_{\mathfrak{b}} \leq D.$$

Proof. As we saw in the proof of Lemma 7.2.2, there exists a positive constant r_0 such that for every $j = 1, \dots, d-1$ and every g as in the statement one has

$$d\left(\Lambda^j U^\tau(g), \Lambda^j_* S^\tau(\sigma^o(g^{-1}))\right) \geq r_0.$$

It is not hard to show (see e.g. Bochi-Potrie-Sambarino [7, Lemma A.7]) that this implies the existence of a constant D such that

$$|a^\tau_j(\sigma^o(g^{-1})g) - a^\tau_j(\sigma^o(g^{-1})) - a^\tau_j(g)| \leq D$$

holds for every $j = 1, \dots, d-1$. By Corollary 6.4.3 we have

$$\left| \tfrac{1}{2} a^\tau_j(\sigma^o(g^{-1})g) - a^\tau_j(g) \right| \leq D/2$$

and we conclude that, up to changing D by larger constant if necessary, one has

$$\left\|\tfrac{1}{2}a^\tau(\sigma^o(g^{-1})g) - a^\tau(g)\right\|_{\mathfrak{b}} \leq D$$

for every g in G as in the statement.

Fix r, ε and L as in Lemma 7.2.2 and Proposition 7.3.2 (for each intersection of C with $\mathsf{F}(V)^o$). By Lemma B.2.2, we can enlarge D if necessary in order to have

$$\left\|\tfrac{1}{2}a^\tau(\sigma^o(g^{-1})g) - \tfrac{1}{2}\lambda(\sigma^o(g^{-1})g)\right\|_{\mathfrak{b}} \leq D$$

for every g as in the statement. To finish apply Proposition 7.3.2 and the fact that the norm $\|\cdot\|_{\mathfrak{b}}$ is W-invariant.

□

Chapter 8

Busemann cocycles and Gromov products

Fix points o in $\mathsf{Q}_{p,q}$, τ in S^o and a Cartan subspace $\mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$. Recall from Subsection 1.6.1 that one has the τ-Busemann cocycle of G:

$$\beta^\tau : \mathsf{G} \times \mathsf{F}(V) \to \mathfrak{b}$$

which is introduced by means of Iwasawa decomposition of G. The goal of this chapter is to introduce an analogue β^o of this cocycle adapted to our setting: in the same way that β^τ "asymptotically controls" the Cartan projection of G, the cocycle β^o "asymptotically controls" the projection b^o. This is done in Section 8.1 while in Section 8.2 we also introduce the analogue of the τ-Gromov product in our setting. Again, as in the Riemannian setting, this analogue "controls" the difference between the (p, q)-Cartan projection of g and its Jordan projection.

8.1 o-Busemann cocycle

Let $\mathfrak{b}^+ \subset \mathfrak{b}$ be a Weyl chamber of the system $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$ and $\mathfrak{a}^+$ be a $\mathfrak{b}^+$-compatible Weyl chamber. For $j = 1, \dots, d-1$ recall that $\chi_j \in \mathfrak{b}^*$ denotes the highest weight of the exterior power representation Λ^j associated to the choice of $\mathfrak{a}^+$.

Set

$$(\mathsf{G} \times \mathsf{F}(V))^o := \{(g, \xi) \in \mathsf{G} \times \mathsf{F}(V)^o : \quad g \cdot \xi \in \mathsf{F}(V)^o\}.$$

Define the o-*Busemann cocycle* of G

$$\beta^o : (\mathsf{G} \times \mathsf{F}(V))^o \to \mathfrak{b}$$

by the equations for $j = 1, \dots, d-1$:

$$\chi_j(\beta^o(g,\xi)) := \frac{1}{2}\log\left|\frac{\langle \Lambda^j g\cdot v, \Lambda^j g\cdot v\rangle_{o_j}}{\langle v,v\rangle_{o_j}}\right|,$$

where $\langle\cdot,\cdot\rangle_{o_j}$ is any form representing o_j and v is any non zero vector in the line $\Lambda^j\xi$. Note that thanks to Lemma 6.4.4 the map β^o is well defined.

The proof of the following is straightforward.

Lemma 8.1.1. *Let g_1 and g_2 be two elements of* G*, ξ be a flag in V and suppose that (g_1g_2,ξ) and (g_2,ξ) belong to $(\mathrm{G}\times\mathsf{F}(V))^o$. Then*

$$\beta^o(g_1g_2,\xi) = \beta^o(g_1, g_2\cdot\xi) + \beta^o(g_2,\xi).$$

□

Remark 8.1.2. The o-Busemann cocycle generalizes the τ-Busemann cocycle of G, in the sense that whenever $pq=0$ and $o=\tau$ one has

$$(\mathrm{G}\times\mathsf{F}(V))^o = \mathrm{G}\times\mathsf{F}(V)$$

and β^o coincides with β^τ (c.f. Quint [56, Lemma 6.4]). On the other hand, since we are assuming $pq\neq 0$, the set $(\mathrm{G}\times\mathsf{F}(V))^o$ does not coincide with $\mathrm{G}\times\mathsf{F}(V)$, but we still have a relation between β^o and β^τ: there exists an smooth function

$$V_{o\tau}:\mathsf{F}(V)^o\to\mathfrak{b}$$

for which one has

$$V_{o\tau}(g\cdot\xi) - V_{o\tau}(\xi) = \beta^o(g,\xi) - \beta^\tau(g,\xi)$$

for every pair $(g,\xi)\in(\mathrm{G}\times\mathsf{F}(V))^o$.

◇

8.1.1 (p,q)-Iwasawa decomposition

The classical relation between Iwasawa decomposition and Busemann cocycles still holds in our setting. To see that, recall that N is the unipotent subgroup of G associated to the choice of $\mathfrak{a}^+$ (Section 1.6). A (p,q)-*Iwasawa decomposition* of an element g in G is a decomposition of the form

$$g = h\hat{w}\exp(X)n$$

where $h\in\mathrm{H}^o$, $\hat{w}\in\hat{\mathrm{W}}$, $X\in\mathfrak{b}$ and $n\in\mathrm{N}$. Note that if this decomposition holds, then the element X is uniquely determined: for every $j=1,\ldots,d-1$ one must have

$$\chi_j(X) = \frac{1}{2}\log\left|\frac{\langle \Lambda^j g\cdot v, \Lambda^j g\cdot v\rangle_{o_j}}{\langle v,v\rangle_{o_j}}\right|,$$

where $\langle\cdot,\cdot\rangle_{o_j}$ is any representative of o_j and v is any non zero vector in the line $\Lambda^j\xi_{\mathfrak{a}^+}$. Indeed, this follows from the fact that $\Lambda^j \mathrm{H}^o$ preserves the form o_j and that the fact that one has the equality

$$|\langle \Lambda^j\hat{w}\cdot v, \Lambda^j\hat{w}\cdot v\rangle_{o_j}| = |\langle v,v\rangle_{o_j}| \tag{8.1.1}$$

for every $j=1,\dots,d-1$ (c.f. Lemma 6.2.3).

On the other hand, if an element $g\in\mathrm{G}$ admits a (p,q)-Iwasawa decomposition then one has

$$(g,\xi_{\mathfrak{a}^+})\in(\mathrm{G}\times\mathsf{F}(V))^o.$$

Conversely, we have the following.

Lemma 8.1.3. *Suppose that the pair $(g,\xi_{\mathfrak{a}^+})$ belongs to $(\mathrm{G}\times\mathsf{F}(V))^o$. Consider an element $\hat{w}\in\hat{\mathrm{W}}$ for which the Weyl chamber $\hat{w}\cdot\mathfrak{a}^+$ is $\mathfrak{b}^+$-compatible and such that $g\cdot\xi_{\mathfrak{a}^+}\in\mathsf{F}(V)^o_{\hat{w}\cdot\mathfrak{a}^+}$. Then g admits a (p,q)-Iwasawa decomposition of the form*

$$g = h\hat{w}\exp(X)n.$$

Proof. There exists $h\in\mathrm{H}^o$ such that $g\cdot\xi_{\mathfrak{a}^+} = h\hat{w}\cdot\xi_{\mathfrak{a}^+}$ and therefore $\hat{w}^{-1}h^{-1}g$ belongs to $\mathrm{P}_{\mathfrak{a}^+}$. Since by Iwasawa decomposition (1.6.1) one has

$$\mathrm{P}_{\mathfrak{a}^+} = \mathrm{M}\exp(\mathfrak{b})\mathrm{N},$$

we conclude that

$$\hat{w}^{-1}h^{-1}g = m\exp(X)n$$

for some $m\in\mathrm{M}$, $X\in\mathfrak{b}$ and $n\in\mathrm{N}$. Now $\hat{w}m = m'\hat{w}$ for some $m'\in\mathrm{M}\subset\mathrm{H}^o$ and the lemma follows.

□

Corollary 8.1.4. *Let (g,ξ) be an element in $(\mathrm{G}\times\mathsf{F}(V))^o$ and take $h'\in\mathrm{H}^o$ and $\hat{w}'\in\hat{\mathrm{W}}$ such that $h'\hat{w}'\cdot\xi_{\mathfrak{a}^+}=\xi$. Then $gh'\hat{w}'$ admits a (p,q)-Iwasawa decomposition of the form*

$$gh'\hat{w}' = h\hat{w}\exp(\beta^o(g,\xi))n.$$

Proof. Since $gh'\hat{w}'\cdot\xi_{\mathfrak{a}^+}$ is o-generic, there exists $\hat{w}\in\hat{\mathrm{W}}$ such that $\hat{w}\cdot\xi_{\mathfrak{a}^+}$ is $\mathfrak{b}^+$-compatible and $gh'\hat{w}'\cdot\xi_{\mathfrak{a}^+}\in\mathsf{F}(V)^o_{\hat{w}\cdot\mathfrak{a}^+}$. By Lemma 8.1.3 we find a (p,q)-Iwasawa decomposition of $gh'\hat{w}'$ of the form

$$gh'\hat{w}' = h\hat{w}\exp(X)n.$$

Now we know that the element X in this decomposition is given by

$$\chi_j(X) := \frac{1}{2}\log\left|\frac{\langle \Lambda^j(gh'\hat{w}')\cdot v, \Lambda^j(gh'\hat{w}')\cdot v\rangle_{o_j}}{\langle v, v\rangle_{o_j}}\right|,$$

where v is any non zero vector in the line $\Lambda^j\xi_{\mathfrak{a}^+}$. Since $\Lambda^j\mathrm{H}^o$ preserves the form o_j, equality (8.1.1) finishes the proof.

□

8.1.2 Dual cocycle

The following is a consequence of Corollary 8.1.4.

Corollary 8.1.5. *Let (g,ξ) be an element in $(\mathrm{G}\times\mathsf{F}(V))^o$. Then $(\sigma^o(g),\xi^{\perp_o})$ belongs to $(\mathrm{G}\times\mathsf{F}(V))^o$ and one has*

$$\beta^o(\sigma^o(g),\xi^{\perp_o}) = \iota_{\mathfrak{a}^+}\circ\beta^o(g,\xi).$$

Proof. Corollary 6.4.2 implies that $(\sigma^o(g),\xi^{\perp_o})$ belongs to $(\mathrm{G}\times\mathsf{F}(V))^o$. Further, let $h'\in\mathrm{H}^o$ and $\hat{w}'\in\hat{\mathrm{W}}$ be two elements such that

$$h'\hat{w}'\cdot\xi_{\mathfrak{a}^+} = \xi.$$

By Corollary 8.1.4 we can write

$$gh'\hat{w}' = h\hat{w}\exp(\beta^o(g,\xi))n$$

and therefore

$$\sigma^o(g) = h\sigma^o(\hat{w})\exp(-\beta^o(g,\xi))\sigma^o(n)\sigma^o(\hat{w}')^{-1}(h')^{-1}.$$

Now by Corollary 6.4.2 we have

$$\sigma^o(\hat{w}')^{-1}(h')^{-1}\cdot(\xi^{\perp_o}) = \xi_{\mathfrak{a}^+}^{\perp_o} = w_{\mathfrak{a}^+}\cdot\xi_{\mathfrak{a}^+}$$

and since $\sigma^o(n)w_{\mathfrak{a}^+} = w_{\mathfrak{a}^+}n'$ for some $n'\in\mathrm{N}$, the result follows.

□

8.2 o-Gromov product

Set

$$(\mathsf{F}(V)^o)^{(2)} := \{(\xi, \xi') \in \mathsf{F}(V)^o \times \mathsf{F}(V)^o : \quad \xi \text{ is transverse to } \xi'\}.$$

Let (ξ, ξ') be an element of $(\mathsf{F}(V)^o)^{(2)}$ and $j = 1, \ldots, d-1$. By Remark 6.4.5 we know that the hyperplane $(\Lambda^j(\xi^{\perp_o}))^{\perp_{o_j}}$ is transverse to the line $\Lambda^j\xi'$. Further, by Lemma 6.4.4 the lines $\Lambda^j(\xi^{\perp_o})$ and $\Lambda^j\xi'$ are not o_j-isotropic. Therefore we can define an *o-Gromov product*

$$\mathbb{G}_o : (\mathsf{F}(V)^o)^{(2)} \to \mathfrak{b}$$

by setting

$$\chi_j(\mathbb{G}_o(\xi, \xi')) := \frac{1}{2} \log \left| \frac{\langle v, v' \rangle_{o_j} \langle v, v' \rangle_{o_j}}{\langle v, v \rangle_{o_j} \langle v', v' \rangle_{o_j}} \right|$$

for every $j = 1, \ldots, d-1$, where v (resp. v') is any non zero vector in the line $\Lambda^j(\xi^{\perp_o})$ (resp. $\Lambda^j\xi'$).

Remark 8.2.1. The map $\mathbb{G}_o$ generalizes that τ-Gromov product $\mathbb{G}_\tau$ of Subsection B.2.1, in the sense that whenever $pq = 0$ and $o = \tau$ one has

$$(\mathsf{F}(V)^o)^{(2)} = \mathsf{F}(V)^{(2)}$$

and $\mathbb{G}_o$ coincides with $\mathbb{G}_\tau$.

$\diamond$

The classical relation between Busemann functions and Gromov products is still satisfied in our framework.

Lemma 8.2.2. *Let $(\xi, \xi') \in (\mathsf{F}(V)^o)^{(2)}$ and $g \in \mathsf{G}$ be an element such that (g, ξ) and (g, ξ') belong to $(\mathsf{G} \times \mathsf{F}(V))^o$. Then the following equality holds:*

$$\mathbb{G}_o(g \cdot \xi, g \cdot \xi') - \mathbb{G}_o(\xi, \xi') = -(\iota_{\mathfrak{a}^+} \circ \beta^o(g, \xi) + \beta^o(g, \xi')).$$

Proof. Fix $j = 1, \ldots, d-1$ and note that, by (6.3.2) and Corollary 6.4.2, if $v \in \Lambda^j(\xi^{\perp_o})$ then

$$\sigma^{o_j}(\Lambda^j g) \cdot v \in \Lambda^j((g \cdot \xi)^{\perp_o}).$$

Let $v' \in \Lambda^j\xi'$ be a non zero vector. We have

$$\begin{aligned}\chi_j(\mathbb{G}_o(g \cdot \xi, g \cdot \xi')) &= \frac{1}{2} \log \left| \frac{\langle \sigma^{o_j}(\Lambda^j g) \cdot v, \Lambda^j g \cdot v' \rangle_{o_j} \langle \sigma^{o_j}(\Lambda^j g) \cdot v, \Lambda^j g \cdot v' \rangle_{o_j}}{\langle \sigma^{o_j}(\Lambda^j g) \cdot v, \sigma^{o_j}(\Lambda^j g) \cdot v \rangle_{o_j} \langle \Lambda^j g \cdot v', \Lambda^j g \cdot v' \rangle_{o_j}} \right| \\ &= \frac{1}{2} \log \left| \frac{\langle v, v' \rangle_{o_j} \langle v, v' \rangle_{o_j}}{\langle \sigma^{o_j}(\Lambda^j g) \cdot v, \sigma^{o_j}(\Lambda^j g) \cdot v \rangle_{o_j} \langle \Lambda^j g \cdot v', \Lambda^j g \cdot v' \rangle_{o_j}} \right|,\end{aligned}$$

where the last equality holds by definition of σ^{o_j}. If we subtract to the previous equality the number

$$\chi_j(\mathbb{G}_o(\xi,\xi')) = \frac{1}{2}\log\left|\frac{\langle v,v'\rangle_{o_j}\langle v,v'\rangle_{o_j}}{\langle v,v\rangle_{o_j}\langle v',v'\rangle_{o_j}}\right|$$

we obtain that $\chi_j(\mathbb{G}_o(g\cdot\xi,g\cdot\xi')) - \chi_j(\mathbb{G}_o(\xi,\xi'))$ equals

$$\frac{1}{2}\log\left|\frac{\langle v,v\rangle_{o_j}\langle v',v'\rangle_{o_j}}{\langle\sigma^{o_j}(\Lambda^j g)\cdot v,\sigma^{o_j}(\Lambda^j g)\cdot v\rangle_{o_j}\langle\Lambda^j g\cdot v',\Lambda^j g\cdot v'\rangle_{o_j}}\right|$$

and by Corollary 8.1.5 the result follows.

□

Recall from Subsection B.2.1 that $\mathbb{B}$ denotes the vector valued cross-ratio. The following gives a geometric interpretation for the o-Gromov product and will be of central importance in Subsection 9.2.2 (c.f. Corollary 9.2.6).

Corollary 8.2.3. *For every $(\xi,\xi')\in(\mathsf{F}(V)^o)^{(2)}$ the following equality holds:*

$$\mathbb{G}_o(\xi,\xi') = -\tfrac{1}{2}\mathbb{B}\left((\xi')^{\perp_o},\xi^{\perp_o},\xi,\xi'\right).$$

Proof. Let $j=1,\dots,d-1$ and $v\in\Lambda^j(\xi^{\perp_o})$ and $v'\in\Lambda^j\xi'$ be non zero vectors. Then by Remark 6.4.5

$$\langle v,\cdot\rangle_{o_j}\in\Lambda^j_*\xi \text{ and } \langle v',\cdot\rangle_{o_j}\in\Lambda^j_*((\xi')^{\perp_o}).$$

We have

$$\mathbb{B}^1(\Lambda^j_*((\xi')^{\perp_o}),\Lambda^j(\xi^{\perp_o}),\Lambda^j_*\xi,\Lambda^j\xi') = \log\left|\frac{\langle v',v'\rangle_{o_j}\langle v,v\rangle_{o_j}}{\langle v',v\rangle_{o_j}\langle v,v'\rangle_{o_j}}\right|$$

and the result is proven.

□

Chapter 9

Counting

We now apply the previous results to the study of the projection $b^o(\rho(\Gamma))$ for a Borel-Anosov representation $\rho : \Gamma \to \mathrm{G}$ and a basepoint o in the set

$$\mathbf{\Omega}_\rho := \{o \in \mathsf{Q}_{p,q} : \quad \xi(\partial_\infty \Gamma) \subset \mathsf{F}(V)^o\},$$

where $\xi : \partial_\infty\Gamma \to \mathsf{F}(V)$ is the limit map of ρ. In Section 9.1 we introduce the counting function

$$t \mapsto \#\{\gamma \in \Gamma : \quad \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t\} \tag{9.0.1}$$

associated to b^o and show that its exponential growth rate is positive, finite and independent on the choice of the basepoint $o \in \mathbf{\Omega}_\rho$ (Corollary 9.1.3). We also describe the asymptotic cone of $b^o(\rho(\Gamma))$ (Proposition 9.1.6). In Section 9.2 we prove a purely exponential asymptotic for a counting function similar to (9.0.1), namely, instead of counting with respect to $\|b^o(\rho\gamma)\|_{\mathfrak{b}}$ we count with respect to $\varphi(b^o(\rho\gamma))$ for carefully chosen linear functionals $\varphi \in \mathfrak{b}^*$ (Proposition 9.2.10). In order to obtain that result we need however an extra assumption: the limit set of ρ must be contained in one connected component of $\mathsf{F}(V)^o$.

9.1 Growth rate and asymptotic cone

9.1.1 Examples

We discuss here some examples of Borel-Anosov representations in G for which the set $\mathbf{\Omega}_\rho$ is non empty.

Example 9.1.1.

- The easiest example comes from a Schottky type construction: one fixes an element $o \in \mathsf{Q}_{p,q}$ and then "ping-pong" constructions provide examples of Borel-Anosov representations whose limit set is contained in $\mathsf{F}(V)^o$.

- Suppose that p and q are different modulo 2 and consider a splitting

$$V = \pi^+ \oplus \pi^-$$

where π^+ (resp. π^-) is a p-dimensional (resp. q-dimensional) subspace of V. Consider the representation

$$\Lambda : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}(V) : \quad \Lambda := \Lambda_p^{\mathrm{irr}} \oplus \Lambda_q^{\mathrm{irr}}$$

where $\Lambda_p^{\mathrm{irr}} : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}(\pi^+)$ and $\Lambda_q^{\mathrm{irr}} : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}(\pi^-)$ are irreducible. Let $\rho_0 : \Gamma_g \to \mathrm{G}$ be a Borel-Anosov representation constructed as in Example 2.1.9 for this choice of Λ.

Pick a quadratic form $o \in \mathsf{Q}_{p,q}$ for which the splitting

$$V = \pi^+ \oplus \pi^-$$

is orthogonal and such that π^+ (resp. π^-) is positive definite (resp. negative definite) for this form. It follows from (2.1.3) that the point o belongs to $\mathbf{\Omega}_{\rho_0}$. Moreover the limit set $\xi(\partial_\infty\Gamma_g)$ is contained in a single open orbit of the action

$$\mathrm{H}^o \curvearrowright \mathsf{F}(V)$$

which is invariant under the map $\cdot^{\perp_o} : \mathsf{F}(V) \to \mathsf{F}(V)$.

- Let $d = 3$, $p = 2$ and $q = 1$ and consider this time a Hitchin representation $\rho : \Gamma_g \to \mathrm{G}$. Suppose that o is a point in $\mathsf{Q}_{2,1}$ such that

  (i) either $\xi^1(x)$ is negative definite for o, for every $x \in \partial_\infty\Gamma_g$,

  or

  (ii) the projective line $\xi^2(x)$ is positive definite for o, for every $x \in \partial_\infty\Gamma_g$,

  then $o \in \mathbf{\Omega}_\rho$. Since the projective limit set $\xi^1(\partial_\infty\Gamma_g)$ is contained in an affine chart of $\mathsf{P}(V)$ (see Choi-Goldman [15]), it is not hard to construct quadratic forms of signature $(2,1)$ on V for which either (i) or (ii) above are satisfied. Conversely, if a point $o \in \mathsf{Q}_{2,1}$ belongs to $\mathbf{\Omega}_\rho$, either condition (i) or two (ii) above must be satisfied (because there exists an affine chart of $\mathsf{P}(V)$ that contains both the projective limit set $\xi^1(\partial_\infty\Gamma)$ and the projectivized isotropic cone of o).

◇

9.1.2 Counting function

Fix a Borel-Anosov representation

$$\rho : \Gamma \to \mathrm{G}$$

and let $\xi = \xi_\rho : \partial_\infty \Gamma \to \mathsf{F}(V)$ be its limit map. Fix points o in $\mathbf{\Omega}_\rho$ and τ in S^o, a Cartan subspace $\mathfrak{a} = \mathfrak{b} \subset \mathfrak{p}^\tau \cap \mathfrak{q}^o$ and a Weyl chamber $\mathfrak{b}^+$ of the system $\Sigma(\mathfrak{g}^{\tau o}, \mathfrak{b})$. The following is a corollary of Lemma 7.2.2.

Corollary 9.1.2. *There exist $0 < \varepsilon_0 \leq r_0$ with the following property: for every $0 < \varepsilon \leq r$ such that $\varepsilon \leq \varepsilon_0$ and $r \leq r_0$, there exists a positive L such that if $\gamma \in \Gamma$ satisfies $|\gamma|_\Gamma > L$ then one has one has that*

$$\sigma^o(\rho\gamma^{-1})\rho\gamma$$

is $(2r, 2\varepsilon)$-loxodromic. In particular one has $\rho\gamma \in \mathscr{B}_{o,\mathrm{G}}$. Further, given a positive δ there number $L > 0$ can be chosen in such a way that

$$d(U^o(\rho\gamma), U^\tau(\rho\gamma)) < \delta \text{ and } d(S^o(\rho\gamma), S^\tau(\rho\gamma)) < \delta.$$

□

Since ρ is projective Anosov it follows that the projective critical exponent δ^1_ρ of ρ is positive and finite: it coincides with the entropy h^1_ρ of ρ (Proposition 2.3.6). It follows from this fact that the *critical exponent*[1]

$$\delta_\rho := \limsup_{t\to\infty} \frac{\log \# \{\gamma \in \Gamma : \quad \|a^\tau(\rho\gamma)\|_{\mathfrak{b}} \leq t\}}{t}$$

of ρ is positive and finite. The following is a consequence of Proposition 7.3.3 and invariance of $\|\cdot\|_{\mathfrak{b}}$ under the action of the Weyl group W.

Corollary 9.1.3. *The function*

$$t \mapsto \# \{\gamma \in \Gamma : \quad \rho\gamma \in \mathscr{B}_{o,\mathrm{G}} \text{ and } \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t\}$$

is finite for every positive t. Moreover, the following equality is satisfied

$$\delta_\rho = \limsup_{t\to\infty} \frac{\log \# \{\gamma \in \Gamma : \quad \rho\gamma \in \mathscr{B}_{o,\mathrm{G}} \text{ and } \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t\}}{t}.$$

In particular, the right hand side of the last equality is finite, positive and independent on the choice of the basepoint $o \in \mathbf{\Omega}_\rho$.

□

[1] For any given Weyl chamber of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$.

9.1.3 Asymptotic cones

Fix a $\mathfrak{b}^+$-compatible Weyl chamber $\tilde{\mathfrak{a}}^+$ for which the intersection

$$\xi(\partial_\infty\Gamma) \cap \mathsf{F}(V)^o_{\tilde{\mathfrak{a}}^+}$$

is non empty and set

$$\mathfrak{a}^+ := \iota_{\mathfrak{b}^+}(\tilde{\mathfrak{a}}^+).$$

Let $a^\tau : \mathrm{G} \to \mathfrak{a}^+$ be the associated Cartan projection. Recall that $\mathscr{L}_\rho$ denotes the asymptotic cone of $\rho(\Gamma)$ (Subsection 2.3.3). We define a new asymptotic cone, that we denote by $\mathscr{L}_\rho^{p,q}$, and that consists on all possible limits of sequences of the form (2.3.5) but with $a^\tau(\rho\gamma_n)$ replaced by $b^o(\rho\gamma_n)$. The main goal of this subsection is to give an explicit description of this new cone by means of Benoist's asymptotic cone (Proposition 9.1.6 below). In order to do that, we set

$$\mathrm{W}_{\rho,\mathfrak{a}^+} := \{w \in \mathrm{W} : \quad w \cdot \mathfrak{a}^+ \subset \mathfrak{b}^+ \text{ and } \xi(\partial_\infty\Gamma) \cap \mathsf{F}(V)^o_{\iota_{\mathfrak{b}^+}(w\cdot\mathfrak{a}^+)} \neq \emptyset\}.$$

Remark 9.1.4. By definition the neutral element 1 belongs to $\mathrm{W}_{\rho,\mathfrak{a}^+}$. Moreover, the equality $\mathrm{W}_{\rho,\mathfrak{a}^+} = \{1\}$ is equivalent to the inclusion

$$\xi(\partial_\infty\Gamma) \subset \mathsf{F}(V)^o_{\tilde{\mathfrak{a}}^+}.$$

◇

Remark 9.1.5. Let $\{\gamma_n\}$ be a sequence in Γ diverging to infinity and suppose that there exists a $\mathfrak{b}^+$-compatible Weyl chamber $\hat{\mathfrak{a}}^+$ such that

$$S^\tau(\rho\gamma_n) \in \mathsf{F}(V)^o_{\hat{\mathfrak{a}}^+}$$

for every n large enough. Let $w \in \mathrm{W}_{\rho,\mathfrak{a}^+}$ be the element defined by the equality

$$w \cdot \mathfrak{a}^+ = \iota_{\mathfrak{b}^+}(\hat{\mathfrak{a}}^+).$$

Then by Proposition 7.3.3 the sequence

$$\|b^o(\rho\gamma_n) - w \cdot a^\tau(\rho\gamma_n)\|_{\mathfrak{b}}$$

is bounded.

◇

Proposition 9.1.6. *The following equality holds:*

$$\mathscr{L}_\rho^{p,q} = \bigcup_{w \in \mathrm{W}_{\rho,\mathfrak{a}^+}} w \cdot \mathscr{L}_\rho.$$

Proof. We first prove the inclusion

$$\mathscr{L}_\rho^{p,q} \subset \bigcup_{w \in \mathrm{W}_{\rho,\mathfrak{a}^+}} w \cdot \mathscr{L}_\rho.$$

Let

$$X = \lim_{n\to\infty} \frac{b^o(\rho\gamma_n)}{t_n}$$

be a point in $\mathscr{L}_\rho^{p,q}$. By taking a subsequence if necessary we may assume

$$S^\tau(\rho\gamma_n) \in \mathsf{F}(V)^o_{\hat{\mathfrak{a}}^+},$$

for all n and some Weyl chamber $\hat{\mathfrak{a}}^+ \subset \mathfrak{b}^+$. Take $w \in \mathrm{W}_{\rho,\mathfrak{a}^+}$ as in Remark 9.1.5. Then the sequence

$$\tfrac{1}{t_n}\|b^o(\rho\gamma_n) - w \cdot a^\tau(\rho\gamma_n)\|_{\mathfrak{b}}$$

converges to zero and we conclude that X belongs to $w \cdot \mathscr{L}_\rho$.

Conversely, let $w \in \mathrm{W}_{\rho,\mathfrak{a}^+}$ and

$$X = \lim_{n\to\infty} \frac{a^\tau(\rho\gamma_n)}{t_n}$$

be an element in $\mathscr{L}_\rho$. Set

$$\hat{\mathfrak{a}}^+ := \iota_{\mathfrak{b}^+}(w \cdot \mathfrak{a}^+),$$

which is $\mathfrak{b}^+$-compatible and, by definition of $\mathrm{W}_{\rho,\mathfrak{a}^+}$, the intersection

$$\xi(\partial_\infty\Gamma) \cap \mathsf{F}(V)^o_{\hat{\mathfrak{a}}^+}$$

is non empty.

By taking a subsequence if necessary, we may suppose that the sequence $\{S^\tau(\rho\gamma_n)\}$ converges to a point $\xi(y)$ as $n \to \infty$. Since the intersection $\xi(\partial_\infty\Gamma) \cap \mathsf{F}(V)^o_{\hat{\mathfrak{a}}^+}$ is non empty we can fix an element γ^0 in Γ such that

$$(\rho\gamma^0)^{-1} \cdot \xi(y) \in \mathsf{F}(V)^o_{\hat{\mathfrak{a}}^+}.$$

Further, we can assume that there exists a constant D such that for every $j = 1, \ldots, d-1$ one has

$$d\left(\Lambda^j U^\tau(\rho\gamma^0), \Lambda^j_* S^\tau(\rho\gamma_n)\right) \geq D$$

for every n large enough. By [7, Lemma A.7] this implies that the sequence

$$\|a^\tau(\rho(\gamma_n\gamma^0)) - a^\tau(\rho\gamma_n)\|_{\mathfrak{b}}$$

is bounded and therefore

$$X = \lim_{n\to\infty} \frac{a^\tau(\rho\gamma_n)}{t_n} = \lim_{n\to\infty} \frac{a^\tau(\rho(\gamma_n\gamma^0))}{t_n}.$$

Thanks to Remark 9.1.5 in order to finish it suffices to show that

$$S^\tau(\rho(\gamma_n\gamma^0))$$

belongs to $\mathsf{F}(V)^o_{\hat{\mathfrak{a}}^+}$ for every n large enough. But applying [7, Lemma A.5] we have

$$\lim_{n\to\infty} S^\tau(\rho(\gamma_n\gamma^0)) = (\rho\gamma^0)^{-1} \cdot \xi_\rho(y)$$

which by construction belongs to $\mathsf{F}(V)^o_{\hat{\mathfrak{a}}^+}$.

□

Remark 9.1.7. Let $\psi_\rho : \mathfrak{b} \to \mathbb{R} \cup \{-\infty\}$ be the *growth indicator* of $\rho(\Gamma)$, introduced by Quint in [55]. For $X \in \mathfrak{b}$, it is defined by

$$\psi_\rho(X) := \|X\|_{\mathfrak{b}} \inf \limsup_{t\to\infty} \frac{\log \#\{\gamma \in \Gamma : \quad a^\tau(\rho\gamma) \in \mathscr{C} \text{ and } \|a^\tau(\rho\gamma)\|_{\mathfrak{b}} \leq t\}}{t},$$

where the infimum is taken over all open cones $\mathscr{C}$ containing X. This function is a central object in the study of asymptotic properties of the Cartan projection a^τ (c.f. Quint [57] and Sambarino [61]). We can define a new growth indicator

$$\psi_\rho^{p,q} : \mathfrak{b} \to \mathbb{R} \cup \{-\infty\}$$

in the same way, using the projection b^o instead of a^τ. Using the same arguments than in Proposition 9.1.6, the following can be proven: let $X \in \mathscr{L}_\rho^{p,q}$ and write $X = w \cdot X_0$ for some $w \in \mathrm{W}_{\rho,\mathfrak{a}^+}$ and $X_0 \in \mathscr{L}_\rho$. Then one has

$$\psi_\rho^{p,q}(X) = \psi_\rho(X_0).$$

Since ψ_ρ equals $-\infty$ outside the asymptotic cone $\mathscr{L}_\rho$, this describes completely the growth indicator $\psi_\rho^{p,q}$. This remark will not be used in the sequel.

◇

9.2 Counting on a given direction in a special case

Through this section we assume further that there exists a single open orbit of the action $\mathrm{H}^o \curvearrowright \mathsf{F}(V)$ that contains the limit set $\xi(\partial_\infty\Gamma)$. The goal of this section is to show that the function

$$t \mapsto \#\{\gamma \in \Gamma : \quad \varphi(b^o(\rho\gamma)) \leq t\}$$

is asymptotically purely exponential as $t \to \infty$, for each $\varphi \in \mathfrak{b}^*$ which is positive in the interior of $\mathscr{L}_\rho^{p,q}$ (Proposition 9.2.10 below).

Recall that $\tilde{\mathfrak{a}}^+$ is a $\mathfrak{b}^+$-compatible Weyl chamber such that $\mathsf{F}(V)^o_{\tilde{\mathfrak{a}}^+}$ contains a point in the limit set of ρ. Therefore our assumption gives the inclusion

$$\xi(\partial_\infty\Gamma) \subset \mathsf{F}(V)^o_{\tilde{\mathfrak{a}}^+}.$$

Recall also that we set $\mathfrak{a}^+ := \iota_{\mathfrak{b}^+}(\tilde{\mathfrak{a}}^+)$ and let $\lambda : \mathrm{G} \to \mathfrak{a}^+$ be the Jordan projection. By Remark 9.1.4 we have

$$\mathrm{W}_{\rho,\mathfrak{a}^+} = \{1\}$$

and Proposition 9.1.6 gives us the equality $\mathscr{L}_\rho^{p,q} = \mathscr{L}_\rho$.

Let $w \in \mathrm{W}$ be the unique element of the Weyl group such that

$$w \cdot \mathfrak{a}^+ = \tilde{\mathfrak{a}}^+.$$

We have the following equalities:

$$w_{\tilde{\mathfrak{a}}^+ \iota_{\mathfrak{b}^+}(\tilde{\mathfrak{a}}^+)} = w \text{ and } w_{\iota_{\mathfrak{b}^+}(\tilde{\mathfrak{a}}^+)\mathfrak{a}^+} = 1.$$

Therefore the following is a consequence of Corollary 7.2.4, Proposition 7.3.2 and Remark 9.1.5.

Corollary 9.2.1. *There exist positive constants L and D such that for every γ in Γ with $|\gamma|_\Gamma > L$ one has*

$$\rho\gamma \in \mathrm{H}^o w \exp\left(\tfrac{1}{2}\lambda(\sigma^o(\rho\gamma^{-1})\rho\gamma)\right) \mathrm{H}^o$$

and

$$\|b^o(\rho\gamma) - a^\tau(\rho\gamma)\|_{\mathfrak{b}} \leq D.$$

□

Example 9.2.2.

- Suppose that ρ is a small deformation of ρ_0, where ρ_0 is as in the second item of Example 9.1.1 and $o \in \mathbf{\Omega}_\rho$ is the point constructed in that example. The Weyl chamber $\tilde{\mathfrak{a}}^+$ satisfies in this case

$$\iota_{\mathfrak{b}^+}(\tilde{\mathfrak{a}}^+) = \tilde{\mathfrak{a}}^+$$

and therefore

$$\rho\gamma \in \mathrm{H}^o \exp\left(\tfrac{1}{2}\lambda(\sigma^o(\rho\gamma^{-1})\rho\gamma)\right)\mathrm{H}^o$$

holds apart from possibly finitely many exceptions γ in Γ_g.

- Suppose that $d = 3$ and $p = 2$. Let ρ be a Hitchin representation and o be a point in $\mathbf{\Omega}_\rho$ (c.f. second item of Example 9.1.1). Then the (p, q)-Cartan decomposition of (large) elements $\rho\gamma$ will always have a non trivial (constant) $\hat{\mathrm{W}}$-coordinate.

$\diamond$

9.2.1 o-Busemann cocycles and o-Gromov product for ρ

Fix a linear functional $\varphi \in \mathfrak{b}^*$ in the interior of $\mathscr{L}^*_\rho$. Set

$$c^\varphi_o : \Gamma \times \partial_\infty\Gamma \to \mathbb{R} : \quad c^\varphi_o(\gamma, x) := \varphi(\beta^o(\rho\gamma, \xi(x)))$$

and

$$\overline{c}^\varphi_o : \Gamma \times \partial_\infty\Gamma \to \mathbb{R} \quad \overline{c}^\varphi_o(\gamma, x) := \varphi(\iota_{\mathfrak{a}^+} \circ \beta^o(\rho\gamma, \xi(x))).$$

These are called the (φ, o)-*Busemann cocycles* of ρ.

The following lemma holds by direct computations.

Lemma 9.2.3. *The pair $(\overline{c}^\varphi_o, c^\varphi_o)$ is a pair of dual Hölder cocycles. The periods of c^φ_o are given by*

$$p_{c^\varphi_o}(\gamma) = \varphi(\lambda(\rho\gamma))$$

for every $\gamma \in \Gamma_{\mathrm{H}}$.

$\square$

Remark 9.2.4. Recall from Subsection 2.3.3 the definition of the pair $(\overline{c}^\varphi_\tau, c^\varphi_\tau)$ of (φ, τ)-Busemann cocycles of ρ. By Remark 8.1.2, the cocycles c^φ_o and c^φ_τ (resp. $\overline{c}^\varphi_o$ and $\overline{c}^\varphi_\tau$) are cohomologous in the sense of Livšic.

$\diamond$

Set

$$[\cdot,\cdot]^{\varphi}_{o} : \partial^2_{\infty}\Gamma \to \mathbb{R} : \quad [x,y]^{\varphi}_{o} := \varphi(\mathbb{G}_o(\xi(x),\xi(y))).$$

The following is a consequence of Lemma 8.2.2.

Corollary 9.2.5. *The map $[\cdot,\cdot]^{\varphi}_{o}$ is a Gromov product for the pair $(\overline{c}^{\varphi}_{o}, c^{\varphi}_{o})$.*

□

We now state a crucial result that will allows us to compare the (p,q)-Cartan projection of an element $\rho\gamma$ with the period $\varphi(\lambda(\rho\gamma))$, by means of the Gromov product. This is the analogue of Lemma 4.3.3 (item *4.*) in the present framework.

Corollary 9.2.6. *Fix a positive δ and A and B two disjoint compact subsets of $\partial_{\infty}\Gamma$. Then there exists a positive L such that for every $\gamma \in \Gamma_{\mathrm{H}}$ satisfying $|\gamma|_{\Gamma} > L$ and $(\gamma_-,\gamma_+) \in A \times B$ one has*

$$|\varphi(b^o(\rho\gamma)) - \varphi(\lambda(\rho\gamma)) + [\gamma_-,\gamma_+]^{\varphi}_{o}| < \delta.$$

Proof. From Corollaries 6.4.3 and 8.2.3 we know that

$$[\gamma_-,\gamma_+]^{\varphi}_{o} = -\tfrac{1}{2}\varphi(\mathbb{B}(\sigma^o(\rho\gamma^{-1})_-, \sigma^o(\rho\gamma^{-1})_+, (\rho\gamma)_-, (\rho\gamma)_+))$$

holds for every $\gamma \in \Gamma_{\mathrm{H}}$. Now observe that because of Corollary 9.2.1 we can suppose that

$$b^o(\rho\gamma) = \tfrac{1}{2}\lambda(\sigma^o(\rho\gamma^{-1})\rho\gamma)$$

holds as well. As in Lemma 4.3.3, now it suffices to apply again Corollary 6.4.3 and Benoist's Theorem B.2.4 to finish the proof.

□

9.2.2 Distribution of (p,q)-Cartan attractors and repellors

In this subsection we assume further that ρ is Zariski dense.

Let μ^{φ}_{o} (resp. $\overline{\mu}^{\varphi}_{o}$) be a Patterson-Sullivan probability of dimension h^{φ}_{ρ} associated to the cocycle c^{φ}_{o} (resp. $\overline{c}^{\varphi}_{o}$): these probabilities exist thanks to the contents of Subsection 2.3.3 and Remark 9.2.4.

Proposition 2.3.10 and Remark 2.3.11 imply the following.

Proposition 9.2.7 (Sambarino [60, Proposition 4.3]). *There exists a positive constant $\tilde{\mathtt{m}}' = \tilde{\mathtt{m}}'_{\rho,o,\varphi}$ such that*

$$\tilde{\mathtt{m}}' e^{-h^{\varphi}_{\rho} t} \sum_{\gamma \in \Gamma_{\mathrm{H}}, \varphi(\lambda(\rho\gamma)) \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \to e^{-h^{\varphi}_{\rho}[\cdot,\cdot]^{\varphi}_{o}} \overline{\mu}^{\varphi}_{o} \otimes \mu^{\varphi}_{o}$$

as $t \to \infty$ *on* $C_c^*(\partial^2_\infty \Gamma)$.

$\square$

As in Section 5.1, the following is an intermediate step towards the proof of Theorem C.

Proposition 9.2.8. *There exists a positive constant* $\tilde{\mathtt{m}}' = \tilde{\mathtt{m}}'_{\rho,o,\varphi}$ *such that*

$$\tilde{\mathtt{m}}' e^{-h^\varphi_\rho t} \sum_{\gamma \in \Gamma_{\mathrm{H}}, \varphi(b^o(\rho\gamma)) \leq t} \delta_{\gamma_-} \otimes \delta_{\gamma_+} \to \overline{\mu}^\varphi_o \otimes \mu^\varphi_o$$

as $t \to \infty$ *on* $C^*(\partial_\infty \Gamma \times \partial_\infty \Gamma)$.

Proof. The proof is the same as that of Proposition 5.1.8. Namely, the convergence outside the diagonal is implied by Corollary 9.2.6. The convergence on the diagonal follows from this one by standard arguments, provided Lemma 9.2.9 below.

$\square$

Lemma 9.2.9 (c.f. Proposition 4.5.1)**.** *Fix an element* $\gamma_0 \in \Gamma$. *Then there exist positive constants* L *and* D_{γ_0} *such that for every* γ *in* Γ *satisfying* $|\gamma|_\Gamma > L$ *one has*

$$\|b^o(\rho(\gamma_0\gamma)) - b^o(\rho\gamma)\|_{\mathfrak{b}} \leq D_{\gamma_0}.$$

Proof. By Corollary 9.2.1 there exist positive constants L and D such that for every γ with $|\gamma|_\Gamma > L$ one has

$$\|b^o(\rho(\gamma_0\gamma)) - b^o(\rho\gamma)\|_{\mathfrak{b}} \leq \|a^\tau(\rho(\gamma_0\gamma)) - a^\tau(\rho\gamma)\|_{\mathfrak{b}} + D.$$

To finish observe that there exists a positive constant d_{γ_0} for which the inequality

$$\|a^\tau(\rho(\gamma_0\gamma)) - a^\tau(\rho\gamma)\|_{\mathfrak{b}} \leq d_{\gamma_0}$$

is satisfied for every $\gamma \in \Gamma$. $\square$

We finally prove Theorem C.

Proposition 9.2.10. *There exists a positive constant* $\tilde{\mathtt{m}}' = \tilde{\mathtt{m}}'_{\rho,o,\varphi}$ *such that*

$$\tilde{\mathtt{m}}' e^{-h^\varphi_\rho t} \sum_{\gamma \in \Gamma, \varphi(b^o(\rho\gamma)) \leq t} \delta_{S^o(\rho\gamma)} \otimes \delta_{U^o(\rho\gamma)} \to \xi_*(\overline{\mu}^\varphi_o) \otimes \xi_*(\mu^\varphi_o)$$

as $t \to \infty$ *on* $C^*(\mathsf{F}(V) \times \mathsf{F}(V))$.

Proof. The proof is analogous to the proof of Proposition 5.1.10, the main steps being:

- Set

$$\nu_t^{\mathrm{H}} := \tilde{\mathtt{m}}' e^{-h_\rho^\varphi t} \sum_{\gamma\in\Gamma_{\mathrm{H}},\varphi(b^o(\rho\gamma))\leq t} \delta_{S^o(\rho\gamma)} \otimes \delta_{U^o(\rho\gamma)}.$$

  Provided with Corollary 9.1.2 and Proposition 2.1.3, the convergence

$$\nu_t^{\mathrm{H}} - \xi_*(\overline{\mu}_o^\varphi) \otimes \xi_*(\mu_o^\varphi) \to 0$$

  follows as in Corollary 5.1.9.

- Set

$$\nu_t := \tilde{\mathtt{m}}' e^{-h_\rho^\varphi t} \sum_{\gamma\in\Gamma,\varphi(b^o(\rho\gamma))\leq t} \delta_{S^o(\rho\gamma)} \otimes \delta_{U^o(\rho\gamma)}$$

  and fix continuous function f on $\mathsf{F}(V)\times\mathsf{F}(V)$ whose support $\operatorname{supp}(f)$ is contained in $\mathsf{F}(V)^{(2)}$. An application of Corollary 9.1.2 and Proposition 2.1.3 yields

$$\#\{\gamma\in\Gamma : \quad (S^o(\rho\gamma), U^o(\rho\gamma)) \in \operatorname{supp}(f) \text{ and } \gamma\notin\Gamma_{\mathrm{H}}\} < \infty.$$

  This implies the convergence $\nu_t^{\mathrm{H}}(f) - \nu_t(f) \to 0$.

- Finally, let $\mathscr{D} := \mathsf{F}(V)\times\mathsf{F}(V)\setminus\mathsf{F}(V)^{(2)}$. Given a positive ε_0 one finds an open covering $\mathscr{U}$ of $\mathscr{D}$ satisfying

$$\limsup_{t\to\infty} \nu_t\left(\bigcup_{U\in\mathscr{U}} U\right) \leq \varepsilon_0$$

  and this finishes the proof.

□

9.2.3 Proof of Corollary D

Recall from Corollary 9.1.3 that we have the equality

$$\delta_\rho = \limsup_{t\to\infty} \frac{\log\#\{\gamma\in\Gamma : \quad \rho\gamma\in\mathscr{B}_{o,\mathrm{G}} \text{ and } \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t\}}{t},$$

where δ_ρ is the critical exponent of ρ.

Corollary 9.2.11. *There exists a strictly positive constant* $\mathtt{C}$ *such that for every* t *large enough one has*

$$\#\{\gamma \in \Gamma : \quad \rho\gamma \in \mathscr{B}_{o,\mathrm{G}} \text{ and } \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t\} \leq \mathtt{C} e^{\delta_\rho t}.$$

Proof. In [55] Quint shows that there exists a form $\Theta_\rho \in \mathfrak{b}^*$ which is strictly positive in the interior of $\mathscr{L}_\rho^{p,q}$, satisfies $\|\Theta_\rho\|_{\mathfrak{b}}^* = \delta_\rho$ and[2]

$$\limsup_{t\to\infty} \frac{\log \#\{\gamma \in \Gamma : \quad \Theta_\rho(a^\tau(\rho\gamma)) \leq t\}}{t} = 1.$$

Because of Corollary 9.2.1 we have

$$\limsup_{t\to\infty} \frac{\log \#\{\gamma \in \Gamma : \quad \rho\gamma \in \mathscr{B}_{o,\mathrm{G}} \text{ and } \Theta_\rho(b^o(\rho\gamma)) \leq t\}}{t} = 1.$$

By Proposition 9.2.10 we conclude that $h_\rho^{\Theta_\rho}$ must be equal to 1.

Now

$$\#\{\gamma \in \Gamma : \quad \rho\gamma \in \mathscr{B}_{o,\mathrm{G}} \text{ and } \|b^o(\rho\gamma)\|_{\mathfrak{b}} \leq t\}$$

is less than or equal to

$$\#\{\gamma \in \Gamma : \quad \rho\gamma \in \mathscr{B}_{o,\mathrm{G}} \text{ and } \Theta_\rho(b^o(\rho\gamma)) \leq \delta_\rho t\}$$

which by Proposition 9.2.10 is equivalent to

$$\frac{e^{\delta_\rho h_\rho^{\Theta_\rho} t}}{\tilde{\mathtt{m}}'_{\rho,o,\Theta_\rho}} = \frac{e^{\delta_\rho t}}{\tilde{\mathtt{m}}'_{\rho,o,\Theta_\rho}}.$$

The corollary then follows.

□

[2]Explicitly, this is the linear functional on $\mathfrak{b}$ that realizes the minimum

$$\min\{\|\varphi\|_{\mathfrak{b}}^* : \quad \varphi \geq \psi_\rho\},$$

where ψ_ρ is the growth indicator of ρ.

Appendix A

Metric Anosov flows

In this appendix we give a quick reminder on the notion of *metric Anosov flows* and its main dynamical features. The goal is, on the one side, to state Bowen's spatial distribution theorem [9, 10] and, on the other, to give a description of the probability of maximal entropy of such a flow more in the spirit of Margulis [42]. All these results are well known and can be found in Pollicott [51] in the setting we need. The exposition will be informal in order to quickly arrive to these results (Facts A.1.1 and A.2.1 below). For complementary information see [51].

A.1 Definition and first properties

Let X be a compact metric space equipped with a continuous flow

$$\phi_t : X \to X.$$

Very informally, the flow ϕ_t is said to be a *metric Anosov flow* if there exist invariant laminations $\mathscr{W}^{ss}$, $\mathscr{W}^{uu}$, $\mathscr{W}^{cs}$ and $\mathscr{W}^{cu}$ of X, called respectively *strong stable lamination*, *strong unstable lamination*, *central stable lamination* and *central unstable lamination*, defining a *local product structure* and with the property that $\mathscr{W}^{ss}$ (resp. $\mathscr{W}^{uu}$) is exponentially contracted by the flow (resp. the inverse flow). For precise definitions see [51].

Recall that ϕ_t is said to be *transitive* if it has a dense orbit and to be *topologically weakly-mixing* if all the periods of its periodic orbits are not multiple of a common constant.

We fix from now on a transitive, topologically weakly-mixing, Hölder continuous metric Anosov flow $\phi_t : X \to X$. As shown by Pollicott [51], the flow admits a Markov coding. We will assume here that this coding is *strong* (c.f. Constantine-Lafont-Thompson [16]). Roughly, this means that the flow can be treated as the suspension of a subshift of finite type and thus the techniques coming from the thermodynamical formalism apply. This has strong dynamical consequences, for instance we have the following.

Fact A.1.1 (c.f. Pollicott [51]). The following holds:

- The topological entropy h_ϕ of ϕ_t is positive and finite. It is given by

$$h_\phi = \lim_{t\to\infty} \frac{\log \#\{a \in \wp : \quad p(a) \leq t\}}{t},$$

  where $\wp$ denotes the set of periodic orbits of ϕ_t and, for $a \in \wp$, the number $p(a)$ denotes the corresponding period.

- As $t \to \infty$, one has

$$h_\phi t e^{-h_\phi t} \#\{a \in \wp : \quad p(a) \leq t\} \to 1.$$

- There exists a unique probability m_ϕ of maximal entropy for ϕ_t, called the *Bowen-Margulis probability* of ϕ_t. This measure is ergodic.

- Periodic orbits become equidistributed with respect to m_ϕ: if Leb_a denotes the Lebesgue measure of length $p(a)$ supported on the periodic orbit a, then

$$h_\phi t e^{-h_\phi t} \sum \frac{1}{p(a)} \mathrm{Leb}_a \to m_\phi$$

  in the weak-star topology as $t \to \infty$. Here the sum is taken over all periodic orbits a for which $p(a) \leq t$.

◇

A.2 The Bowen-Margulis measure

As shown by Margulis [42], for Anosov flows there exists a measure on local central unstable leaves which is exponentially contracted by the flow[1]. By reversing time and disintegrating along flow lines one finds a family of measures on local strong stable leaves which is expanded by the flow. Margulis [42] first showed how to combine these families to produce a finite invariant Borel measure in the whole space which is, up to scaling, the Bowen-Margulis probability of the flow.

In the following fact we briefly describe this process. A proof in the Anosov flow case can be found in Katok-Hasselblatt's book [34, Section 5 of Chapter 20]. This proof equally applies in our setting.

[1]In our context this measure is also available: this follows from the thermodynamical formalism as explained by Bowen-Marcus [11, Section 4].

Fact A.2.1. Suppose that there exists a real number $\delta^u \geq 0$ such that for every $x_0 \in X$ and any small relative neighbourhood $\mathscr{W}^{cu}_{\text{loc}}(x_0)$ of x_0 in $\mathscr{W}^{cu}(x_0)$ there exists a positive and finite Borel measure $\nu^{cu}_{\text{loc}}(x_0)$ on $\mathscr{W}^{cu}_{\text{loc}}(x_0)$ for which the equality

$$(\phi_t)_* \nu^{cu}_{\text{loc}}(x_0) = e^{-\delta^u t} \nu^{cu}_{\text{loc}}(\phi_t(x_0))$$

holds for every $t \in \mathbb{R}$.

Further, suppose that one has a real number $\delta^s \geq 0$ and a family of measures $\{\nu^{ss}_{\text{loc}}(x_0)\}$ on local leaves of $\mathscr{W}^{ss}$ for which

$$(\phi_t)_* \nu^{ss}_{\text{loc}}(x_0) = e^{\delta^s t} \nu^{ss}_{\text{loc}}(\phi_t(x_0)) \tag{A.2.1}$$

is satisfied for every $t \in \mathbb{R}$. Moreover, suppose that the family $\{\nu^{ss}_{\text{loc}}(x_0)\}$ is a *transverse measure*[2] *of class C^0 for $\mathscr{W}^{cu}$*. Consider the measure on X locally given by

$$m(A) := \int_{x \in \mathscr{W}^{cu}_{\text{loc}}(x_0)} \nu^{ss}_{\text{loc}}(x)(A \cap \mathscr{W}^{ss}_{\text{loc}}(x)) d\nu^{cu}_{\text{loc}}(x_0)(x).$$

Then one has the equalities $\delta^u = \delta^s = h_\phi$, the measure m is ϕ_t-invariant and it is proportional to the Bowen-Margulis probability m_ϕ.

◇

The following remark is sometimes useful.

Remark A.2.2. Let $\mathscr{F}$ be a lamination on X that determines a local product structure with $\mathscr{W}^{cu}$. Further, let $\{\nu^{\mathscr{F}}_{\text{loc}}(x)\}_{x \in X}$ be a transverse measure of class C^0 for $\mathscr{W}^{cu}$ supported on local leaves of $\mathscr{F}$. It makes sense to ask if equation (A.2.1) is satisfied for this family: one can uniquely extend $\{\nu^{\mathscr{F}}_{\text{loc}}(x)\}_{x \in X}$ to a familiy $\{\nu_T\}_{T \in \mathscr{T}}$, where $\mathscr{T}$ is the set of transversals to $\mathscr{W}^{cu}$, and condition (A.2.1) makes complete sense for the family $\{\nu_T\}_{T \in \mathscr{T}}$ (see [38, p.109]).

Suppose then that $\{\nu^{\mathscr{F}}_{\text{loc}}(x)\}_{x \in X}$ is a transverse measure of class C^0 for $\mathscr{W}^{cu}$ supported on local leaves of $\mathscr{F}$ that satisfies equation (A.2.1) and consider the measure on X locally given by

$$m'(A) := \int_{x \in \mathscr{W}^{cu}_{\text{loc}}(x_0)} \nu^{\mathscr{F}}_{\text{loc}}(x)(A \cap \mathscr{F}_{\text{loc}}(x)) d\nu^{cu}_{\text{loc}}(x_0)(x).$$

Since locally one has the equality

[2] For a precise definition see Ledrappier [38, p.109]. Very informally, this means that if we have a map between two local leaves $\mathscr{W}^{ss}_{\text{loc}}(x_0)$ and $\mathscr{W}^{ss}_{\text{loc}}(x_1)$ of $\mathscr{W}^{ss}$ which is defined "following the leaves of $\mathscr{W}^{cu}$", then this map sends the measure $\nu^{ss}_{\text{loc}}(x_0)$ to a measure which is absolutely continuous with respect to the measure $\nu^{ss}_{\text{loc}}(x_1)$.

$$(\phi_t)_* m' = e^{(\delta^s - \delta^u)t} m',$$

this condition must also be global and therefore $\delta^u = \delta^s$ and the measure m' is ϕ_t-invariant. Further, it is absolutely continuous with respect to m, which is ergodic because the Bowen-Margulis probability is. We conclude that m' must be proportional to m_ϕ.

◇

A.3 Reparametrizations

Let $f : X \to \mathbb{R}$ be a strictly positive continuous function. The *reparametrization* of ϕ_t by f is the flow

$$\phi_t^f : X \to X$$

characterized by the formula

$$\phi_t(x) = \phi^f_{\int_0^t f\phi_s(x)ds}(x)$$

for every $x \in X$ and every $t \in \mathbb{R}$ (see Sambarino [60, Section 2] for a precise definition). If the function f is Hölder continuous the flow ϕ_t^f is said to be a *Hölder reparametrization* of ϕ_t.

Note that if ϕ_t^f is a reparametrization of ϕ_t then $\wp$ is still the set of periodic orbits of ϕ_t^f. If $a \in \wp$, the corresponding period for ϕ_t^f is given by

$$\int_0^{p(a)} f\phi_t(x)dt,$$

where x is any point in a.

Proposition A.3.1. *Let ϕ_t^f be a Hölder reparametrization of ϕ_t. Then ϕ_t^f is a transitive metric Anosov flow with the same central unstable and central stable laminations than ϕ_t. Further, ϕ_t^f admits a strong Markov coding.*

Proof. The strong stable lamination of ϕ_t^f can be computed as follows. Let $x_0 \in X$ be a point. For every $x \in \mathscr{W}^{ss}(x_0)$ we let

$$T_{x_0}(x) := \lim_{t\to\infty} \int_0^t f\phi_s(x)ds - \int_0^t f\phi_s(x_0)ds.$$

Then the strong stable leaf of ϕ_t^f through the point x_0 is given by

$$\mathscr{W}_f^{ss}(x_0) := \{\phi_{T_{x_0}(x)}(x) : \quad x \in \mathscr{W}^{ss}(x_0)\}.$$

For the existence of a strong Markov coding for ϕ_t^f see [60, Lemma 2.9]. □

Appendix B

Products of proximal elements

In Subsection 1.5.2 the notion of F_θ-proximality and (r, ε)-proximality was introduced for a general semisimple Lie group with finite center and no compact factors. In this appendix we give a finer analysis of this notion for the special case $\mathrm{G} = \mathrm{PSL}(V)$. We are interested notably in estimates of the Cartan and Jordan projection of products of (r, ε)-proximal elements in $\mathsf{P}(V)$ and in $\mathsf{F}(V)$ (Theorems B.1.5 and B.2.4 below). These results are well known but we provide proofs for those not explicitly stated in the literature. Standard references for this part are the works of Benoist [3, 4, 5].

B.1 Proximal elements in $\mathsf{P}(V)$

B.1.1 Notations

Recall that for $j = 1, \ldots, d$ the symbol

$$\mathsf{Gr}_j(V)$$

denotes the Grassmannian of j-dimensional subspaces of V and that for $j = 1$ we use the special notation $\mathsf{P}(V) := \mathsf{Gr}_1(V)$. There exists a G-equivariant identification $\mathsf{P}(V^*) \to \mathsf{Gr}_{d-1}(V)$ given by

$$\vartheta \mapsto \ker \vartheta$$

where the action of G on the left side is given by $g \cdot \vartheta := \vartheta \circ g^{-1}$. This identification will be used whenever convenient.

A Euclidean norm $\| \cdot \|$ on V will be fixed in the whole appendix. For $\xi_1, \xi_2 \in \mathsf{P}(V)$ define the (continuous) distance

$$d(\xi_1, \xi_2) := \inf\{\|v_{\xi_1} - v_{\xi_2}\| : \quad v_{\xi_i} \in \xi_i \text{ and } \|v_{\xi_i}\| = 1 \text{ for all } i = 1, 2\}.$$

For $\eta_1, \eta_2 \in \mathsf{Gr}_{d-1}(V)$, we denote by $d^*(\eta_1, \eta_2)$ the distance on $\mathsf{P}(V^*)$ induced by the operator norm $\|\cdot\|^*$ on V^*.

On the other hand, let

$$\mathsf{P}(V)^{(2)} := \{(\vartheta, v) \in \mathsf{P}(V^*) \times \mathsf{P}(V) : \quad v \notin \ker \vartheta\},$$

that is, $\mathsf{P}(V)^{(2)} = (\mathsf{P}(V^*) \times \mathsf{P}(V))^{(2)}$. Also, let

$$\mathsf{P}(V)^{(4)} := \{(\vartheta, v, \phi, u) \in \mathsf{P}(V)^{(2)} \times \mathsf{P}(V)^{(2)} : \quad v \notin \ker \phi \text{ and } u \notin \ker \vartheta\}.$$

Observe that

$$\mathbb{G}^1_{\|\cdot\|} = \mathbb{G}^1 : \mathsf{P}(V)^{(2)} \to \mathbb{R} : \quad \mathbb{G}^1(\vartheta, v) := \log \frac{|\vartheta(v)|}{\|\vartheta\|^* \|v\|} \tag{B.1.1}$$

is well defined. It is called the *projective Gromov product* of the pair (ϑ, v) (c.f. Sambarino [61, p. 1780]). Note that if $\|\cdot\|'$ is a Euclidean norm on V proportional to $\|\cdot\|$, then one has

$$\mathbb{G}^1_{\|\cdot\|} = \mathbb{G}^1_{\|\cdot\|'}.$$

Because of this, if $\tau \in \mathsf{X}_{\mathrm{G}}$ is the Cartan involution of G associated to $\|\cdot\|$, we will sometimes use the notation $\mathbb{G}^1_\tau$ for the projective Gromov product and call it the *projective τ-Gromov product.*

Similarly, following Benoist [5, p. 3] we let

$$\mathbb{B}^1 : \mathsf{P}(V)^{(4)} \to \mathbb{R} : \quad \mathbb{B}^1(\vartheta, v, \phi, u) := \log \left| \frac{\vartheta(u)}{\vartheta(v)} \frac{\phi(v)}{\phi(u)} \right|. \tag{B.1.2}$$

This map is called de *projective cross-ratio* of (ϑ, v, ϕ, u)[1]. Both $\mathbb{G}^1$ and $\mathbb{B}^1$ are continuous.

B.1.2 Product of proximal elements

Given g in $\mathfrak{gl}(V) \setminus \{0\}$ we denote by

$$\lambda_1(g) \geq \cdots \geq \lambda_d(g)$$

the logarithms of the moduli of the eigenvalues of g, repeated with multiplicity (we use the convention $\log 0 = -\infty$). The element g is said to be *proximal* in $\mathsf{P}(V)$ if $\lambda_1(g)$ is simple[2]. In that case we let g_+ (resp. g_-) to be the attractive fixed line (resp. repelling fixed hyperplane) of g in $\mathsf{P}(V)$. Note that if g is non invertible then g_- contains the kernel of g. Let $0 < \varepsilon \leq r$. The notion of (r, ε)*-proximality* in $\mathsf{P}(V)$ for elements in $\mathfrak{gl}(V) \setminus \{0\}$ is analogous to the one of Subsection 1.5.2.

---

[1] Sometimes $e^{\mathbb{B}^1}$ is called the projective cross-ratio.

[2] Note that this definition is consistent with the one introduced in Subsection 1.5.2.

Lemma B.1.1 (Benoist [3, Corollaire 6.3]). *Let $0 < \varepsilon \leq r$. There exists a constant $c_{r,\varepsilon} > 0$ such that for every element g which is (r,ε)-proximal in $\mathsf{P}(V)$ one has*

$$\log \|g\| - c_{r,\varepsilon} \leq \lambda_1(g) \leq \log \|g\|.$$

□

The following criterion of (r,ε)-proximality is useful.

Lemma B.1.2 (Benoist [3, Lemme 6.2]). *Let g be an element in $\mathfrak{gl}(V) \setminus \{0\}$, $\eta \in \mathsf{Gr}_{d-1}(V)$, $\xi \in \mathsf{P}(V)$ and $0 < \varepsilon \leq r$. If $d(\xi,\eta) \geq 6r$ and $g \cdot B_\varepsilon(\eta) \subset b_\varepsilon(\xi)$ then g is $(2r, 2\varepsilon)$-proximal in $\mathsf{P}(V)$ with*

$$d(g_+, \xi) \leq \varepsilon \text{ and } d^*(g_-, \eta) \leq \varepsilon.$$

Proof. Consider the Hilbert distance on the convex set $B_\varepsilon(\eta)$ (see [6]). The condition $g \cdot B_\varepsilon(\eta) \subset b_\varepsilon(\xi)$ implies that g is contracting for this metric and thus has a unique fixed point in $B_\varepsilon(\eta)$, which belongs in fact to $b_\varepsilon(\xi)$. The proof now finishes as in [3, Lemme 6.2].

□

Corollary B.1.3 (Benoist [5, Lemme 1.4]). *Let $0 < \varepsilon \leq r$. If g_1 and g_2 are (r,ε)-proximal in $\mathsf{P}(V)$ and satisfy*

$$d({g_1}_+, {g_2}_-) \geq 6r \text{ and } d({g_2}_+, {g_1}_-) \geq 6r$$

then $g_1 g_2$ is $(2r, 2\varepsilon)$-proximal in $\mathsf{P}(V)$.

□

Let g_1 and g_2 be two matrices as in Corollary B.1.3. The goal now is to state a theorem (Theorem B.1.5) which provides a comparison between the spectral radius and operator norm of $g_1 g_2$ in terms of the spectral radii of g_1 and g_2 and the projective Gromov product and projective cross-ratio.

Lemma B.1.4. *Fix $r > 0$ and $\delta > 0$. For every ε small enough, the following property is satisfied: for every pair g_1 and g_2 of (r,ε)-proximal elements in $\mathsf{P}(V)$ such that*

$$d({g_1}_+, {g_2}_-) \geq 6r \text{ and } d({g_2}_+, {g_1}_-) \geq 6r$$

one has

$$|\mathbb{G}^1({g_2}_-, {g_1}_+) - \mathbb{G}^1((g_1 g_2)_-, (g_1 g_2)_+)| < \delta.$$

Proof. For every $0 < \varepsilon \leq r$, consider the compact set $C_{r,\varepsilon}$ of pairs (g_1, g_2) of norm one (r,ε)-proximal elements in $\mathsf{P}(V)$ belonging to $\mathfrak{gl}(V) \setminus \{0\}$ and satisfying

$$d(g_{1_+}, g_{2_-}) \geq 6r \text{ and } d(g_{2_+}, g_{1_-}) \geq 6r.$$

The function

$$(g_1, g_2) \mapsto |\mathbb{G}^1(g_{2_-}, g_{1_+}) - \mathbb{G}^1((g_1g_2)_-, (g_1g_2)_+)|$$

is continuous and equals zero on $C_r := \cap_{\varepsilon>0} C_{r,\varepsilon} \subset \mathfrak{gl}(V) \setminus \{0\}$.

□

Theorem B.1.5 (Benoist [5, Lemme 1.4]). *Fix $r > 0$ and $\delta > 0$. Then for every ε small enough, the following properties are satisfied: for every pair g_1 and g_2 of (r, ε)-proximal elements in $\mathsf{P}(V)$ such that*

$$d(g_{1_+}, g_{2_-}) \geq 6r \text{ and } d(g_{2_+}, g_{1_-}) \geq 6r$$

one has:

1. *The number*

$$\left|\lambda_1(g_1g_2) - (\lambda_1(g_1) + \lambda_1(g_2)) - \mathbb{B}^1(g_{1_-}, g_{1_+}, g_{2_-}, g_{2_+})\right|$$

   is less than δ.

2. *The number*

$$\left|\log \|g_1g_2\| - (\lambda_1(g_1) + \lambda_1(g_2)) - \mathbb{B}^1(g_{1_-}, g_{1_+}, g_{2_-}, g_{2_+}) + \mathbb{G}^1_{\|\cdot\|}(g_{2_-}, g_{1_+})\right|$$

   is less than δ.

Proof. 1. See [5, Lemme 1.4].

2. Let ε be as in (1). For every g_1 and g_2 as in the statement, Corollary B.1.3 implies that g_1g_2 is $(2r, 2\varepsilon)$-proximal in $\mathsf{P}(V)$. By [60, Lemma 5.6] (and taking ε smaller if necessary) we have

$$\left|\log \|g_1g_2\| - \lambda_1(g_1g_2) + \mathbb{G}^1_{\|\cdot\|}((g_1g_2)_-, (g_1g_2)_+)\right| < \delta.$$

   Lemma B.1.4 finishes the proof.

□

B.2 Loxodromic elements

B.2.1 Notations

For $j = 1, \dots, d$ let $\Lambda^j V$ be the j^{th}-exterior power representation of G. Recall that there exists an equivariant map

$$\Lambda^j : \mathsf{F}(V) \to \mathsf{P}(\Lambda^j V)$$

defined in the following way. For a full flag $\xi = (\xi^1, \dots, \xi^d) \in \mathsf{F}(V)$ take an ordered basis $\{v_1, \dots, v_d\}$ of V such that for every $i = 1, \dots, d$ one has

$$\xi^i = \operatorname{span}\{v_1, \dots, v_i\}.$$

Then $\Lambda^j \xi$ is, by definition, the line spanned by $v_1 \wedge \cdots \wedge v_j$. Also we have an equivariant map

$$\Lambda^j_* : \mathsf{F}(V) \to \mathsf{P}((\Lambda^j V)^*),$$

which is induced by the representation dual to Λ^j, and that can be computed as follows: $\Lambda^j_* \xi$ is the class of a functional on $\Lambda^j V$ whose kernel coincides with

$$\operatorname{span}\{v_{i_1} \wedge \cdots \wedge v_{i_j} : \quad i_1 < \cdots < i_j \text{ and } i_k \neq d + k - j \text{ for all } 1 \leq k \leq j\}.$$

Remark B.2.1. Two full flags ξ_1 and ξ_2 of V are transverse if and only if the line $\Lambda^j \xi_1$ of $\Lambda^j V$ is transverse to the hyperplane $\Lambda^j_* \xi_2$, for every $j = 1, \dots, d-1$.

◇

The norm $\|\cdot\|$ induces a Euclidean norm $\|\cdot\|_j$ on $\Lambda^j V$. For elements $\xi_1, \xi_2 \in \mathsf{F}(V)$, set

$$d^{\mathsf{F(V)}}_{\|\cdot\|}(\xi_1, \xi_2) := \max_{j=1,\dots,d-1} d_{\|\cdot\|_j}(\Lambda^j \xi_1, \Lambda^j \xi_2).$$

Then $d^{\mathsf{F(V)}}_{\|\cdot\|}(\cdot, \cdot)$ defines a continuous distance on $\mathsf{F}(V)$. It will be denoted by $d(\cdot, \cdot)$ whenever there is no risk of confusion.

Let τ be the point in X_{G} associated to the Euclidean norm $\|\cdot\|$ and fix a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}^\tau$ and a closed Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$. For $j = 1, \dots, d-1$ denote by $\chi_j \in \mathfrak{a}^*$ the *highest weight* of the exterior power representation Λ^j associated to the choice of $\mathfrak{a}^+$. Explicitely one has

$$\chi_j = \sum_{i=1}^{j} \varepsilon_i$$

where, for $i = 1, \ldots, d$, the functional $\varepsilon_i \in \mathfrak{a}^*$ is the one introduced in Example 1.4.2. The map

$$X \mapsto (\chi_1(X), \ldots, \chi_{d-1}(X))$$

is a linear isomorphism between $\mathfrak{a}$ and $\mathbb{R}^{d-1}$.

Recall that $\mathsf{F}(V)^{(2)}$ denotes the set of ordered pairs of transverse full flags of V and let $\mathsf{F}(V)^{(4)}$ denote the set of elements of the form

$$(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathsf{F}(V)^4$$

such that (ξ_i, ξ_k) belongs to $\mathsf{F}(V)^{(2)}$ for all $(i, k) \in \{(1, 2), (1, 4), (2, 3), (3, 4)\}\}$. Define the *vector valued τ-Gromov product*

$$\mathbb{G}_\tau = \mathbb{G} : \mathsf{F}(V)^{(2)} \to \mathfrak{a} \tag{B.2.1}$$

by letting

$$\chi_j(\mathbb{G}(\xi_1, \xi_2)) := \mathbb{G}^1(\Lambda^j_* \xi_1, \Lambda^j \xi_2)$$

for every $j = 1, \ldots, d-1$. Similarly, the *vector valued cross-ratio*

$$\mathbb{B} : \mathsf{F}(V)^{(4)} \to \mathfrak{a} \tag{B.2.2}$$

is defined by the equalities

$$\chi_j(\mathbb{B}(\xi_1, \xi_2, \xi_3, \xi_4)) := \mathbb{B}^1(\Lambda^j_* \xi_1, \Lambda^j \xi_2, \Lambda^j_* \xi_3, \Lambda^j \xi_4)$$

for every $j = 1, \ldots, d-1$.

B.2.2 Product of loxodromic elements

An element g in G is said to be *loxodromic* (resp. (r, ε)*-loxodromic*) if $\Lambda^j g$ is proximal (resp. (r, ε)-proximal) in $\mathsf{P}(\Lambda^j V)$ for every $j = 1, \ldots, d-1$. Note that the following are equivalent:

- The element g is loxodromic.
- The element g is proximal in $\mathsf{F}(V)$.
- The vector $\lambda(g)$ belongs to the interior $\mathrm{int}(\mathfrak{a}^+)$ of $\mathfrak{a}^+$.
- Any lift of g to $\mathrm{SL}(V)$ is diagonalizable, with eigenvalues of different moduli.
- There exists a unique basis of lines $\mathcal{C}_g$ of V whose elements are fixed by g.

If $g \in \mathrm{G}$ is loxodromic, the elements of $\mathcal{C}_g$ are said to be the *eigenlines* of g and we say that $\mathcal{C}_g$ *diagonalizes* the element g (and that g is *diagonalizable*). Also, note that in this case the line $\Lambda^j(g_+)$ (resp. the hyperplane $\Lambda^j_*(g_-)$) is the attractive line (resp. repelling hyperplane) of $\Lambda^j g$ acting on $\mathsf{P}(\Lambda^j V)$, for every $j = 1, \dots, d-1$.

The following results are direct consequence of the contents of Subsection B.1.2.

Lemma B.2.2. *Let $0 < \varepsilon \leq r$. There exists a constant $c_{r,\varepsilon} > 0$ such that for every element g which is (r, ε)-loxodromic one has*

$$\|a^\tau(g) - \lambda(g)\|_{\mathfrak{a}} \leq c_{r,\varepsilon}.$$

□

Lemma B.2.3. *Let g be an element in $\mathfrak{gl}(V) \setminus \{0\}$, ξ_1 and ξ_2 be two flags in V and $0 < \varepsilon \leq r$. Suppose that for every $j = 1, \dots, d-1$ one has*

$$d(\Lambda^j \xi_1, \Lambda^j_* \xi_2) \geq 6r$$

and

$$\Lambda^j g \cdot B_\varepsilon(\Lambda^j_* \xi_2) \subset b_\varepsilon(\Lambda^j \xi_1).$$

Then g is $(2r, 2\varepsilon)$-loxodromic with

$$d(g_+, \xi_1) \leq \varepsilon \text{ and } d(g_-, \xi_2) \leq \varepsilon.$$

□

Theorem B.2.4 (Benoist [5, Lemme 3.4])**.** *Fix $r > 0$ and $\delta > 0$. Then for every ε small enough, the following properties are satisfied: for every pair of (r, ε)-loxodromic elements g_1 and g_2 such that for every $j = 1, \dots, d-1$ one has*

$$d(\Lambda^j(g_{1+}), \Lambda^j_*(g_{2-})) \geq 6r \text{ and } d(\Lambda^j(g_{2+}), \Lambda^j_*(g_{1-})) \geq 6r,$$

the product $g_1 g_2$ is $(2r, 2\varepsilon)$-loxodromic. Further:

1. *The number*

$$\left\|\lambda(g_1 g_2) - (\lambda(g_1) + \lambda(g_2)) - \mathbb{B}(g_{1-}, g_{1+}, g_{2-}, g_{2+})\right\|_{\mathfrak{a}}$$

is less than δ.

2. *The number*

$$\left\|a^\tau(g_1 g_2) - (\lambda(g_1) + \lambda(g_2)) - \mathbb{B}(g_{1-}, g_{1+}, g_{2-}, g_{2+}) + \mathbb{G}_\tau(g_{2-}, g_{1+})\right\|_{\mathfrak{a}}$$

is less than δ.

□

Appendix C

Domains of discontinuity in some homogeneous spaces

What we present here is part of an still ongoing project joint with Florian Stecker. The exposition follows closely the one presented by Stecker in his Ph.D Thesis [63]. The reader is referred to [63] for a more complete and detailed account on the subject.

The chapter contains two sections. In Section C.1 we introduce the notion of *fat ideal* generalizing that of Kapovich-Leeb-Porti [30]. In Section C.2 we show that the construction of [30], associating a domain of discontinuity to each Anosov representation and each fat ideal, still holds in our more general setting.

C.1 Relative positions and fat ideals

This section is structured as follows. In Subsection C.1.2 we define the set of *relative positions* and a partial order in it. We also provide a topological characterization of maximal positions that will be useful in Subsection C.2.3 (Lemma C.1.3 below). In Subsection C.1.3 we introduce the concept of *fat ideal* in our setting.

C.1.1 Relative positions

Given a Lie group G and two closed subgroups H_1 and H_2 of G, we call the double quotient $\mathrm{H}_1\backslash\mathrm{G}/\mathrm{H}_2$ the set of *relative positions* between points in G/H_1 and G/H_2. The *relative position map* is

$$\mathbf{pos}: \mathrm{G}/\mathrm{H}_1 \times \mathrm{G}/\mathrm{H}_2 \to \mathrm{H}_1\backslash\mathrm{G}/\mathrm{H}_2 : \quad \mathbf{pos}(g_1\mathrm{H}_1, g_2\mathrm{H}_2) := \mathrm{H}_1 g_1^{-1} g_2 \mathrm{H}_2.$$

Note that G acts on $\mathrm{G}/\mathrm{H}_1 \times \mathrm{G}/\mathrm{H}_2$ diagonally and that **pos** is G-invariant. Conversely, we have the following remark.

Remark C.1.1. Let x and x' be two points in G/H_1 and o and o' be two points in G/H_2 such that

$$\mathbf{pos}(x, o) = \mathbf{pos}(x', o').$$

Then there exists an element $g \in \mathrm{G}$ for which one has

$$g \cdot (x, o) = (x', o').$$

◇

C.1.2 Partial order and maximal positions

We suppose from now on that H_1 and H_2 are chosen in such a way that the associated set of relative positions is finite. Two interesting classes of examples that satisfy this hypothesis are the following:

- The case in which G is semisimple with finite center and both H_1 and H_2 are parabolic subgroups. Finiteness of the set of relative positions follows in this case from the *Bruhat decomposition* of G (see Knapp [35, Theorem 7.40]).
- The case in which G is semisimple with finite center, H_1 is a parabolic subgroup and H_2 is the fixed point subgroup of an involutive automorphism of G (see Wolf [65]).

Given two elements g and g' in G, the notation $\mathrm{H}_1 g \mathrm{H}_2 \leq \mathrm{H}_1 g' \mathrm{H}_2$ stands for

$$\mathrm{H}_1 g \mathrm{H}_2 \subset \overline{\mathrm{H}_1 g' \mathrm{H}_2}.$$

The finiteness assumption on $\mathrm{H}_1 \backslash \mathrm{G} / \mathrm{H}_2$ implies that the relation $\leq$ is a partial order in the set of relative positions (see Stecker [63, Lemma 2.1.3]).

Example C.1.2. Let V be a real vector space of dimension 3 and $\mathrm{G} := \mathrm{PSL}(V)$. In Figure C.1 we illustrate the partial order $\leq$ in the following two cases:

- The case in which $\mathrm{H}_1 = \mathrm{H}_2 = \mathrm{P}_{\mathfrak{a}^+}$ is a minimal parabolic subgroup of G.
- The case in which $\mathrm{H}_1 = \mathrm{P}_{\mathfrak{a}^+}$ and $\mathrm{H}_2 = \mathrm{PSO}(2, 1)$ is the stabilizer of a point $o \in \mathsf{Q}_{2,1}$.

◇

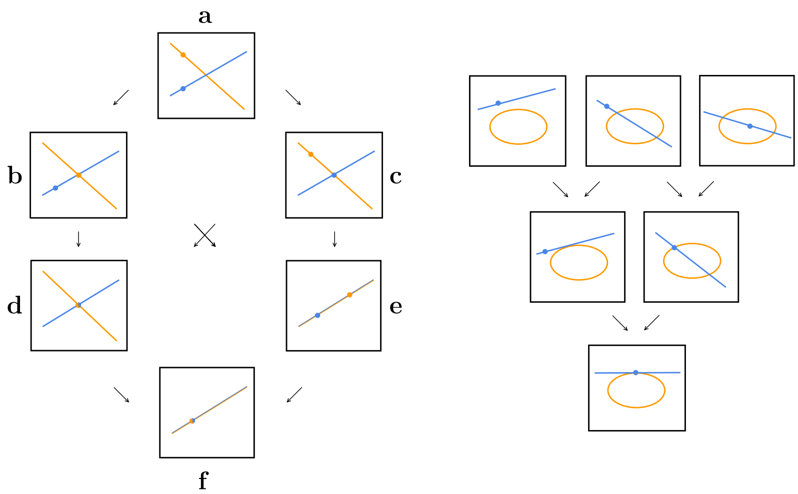


Figure C.1: The partial order $\leq$ in two cases. Light blue objects are points in G/H_1 while orange objects correspond to points in G/H_2. On the left, the case corresponding to the first item of Example C.1.2. On the right, the case corresponding to the second item (what is drawn in orange is the projectivized isotropic cone of a point in $\mathsf{Q}_{2,1}$). The arrows stand for the $\leq$ relation. For instance, positions $\mathbf{a}$ and $\mathbf{b}$ on the left are related by $\mathbf{b} \leq \mathbf{a}$.

Even though is not formally needed to introduce the notion of fat ideal, we finish this subsection with a topological characterization for a relative position being maximal.

Two elements $x \in \mathrm{G}/\mathrm{H}_1$ and $o \in \mathrm{G}/\mathrm{H}_2$ for which

$$\mathbf{pos}(x, o)$$

is maximal are said to be *transverse*. Because of the following lemma, this definition is consistent with the notion of transverse flags given in Section 1.5 of Chapter 1.

Lemma C.1.3. *Let o be a point in G/H_2. Then a relative position $\mathbf{p}$ is maximal if and only if the set*

$$\{x \in \mathrm{G}/\mathrm{H}_1 : \quad \mathbf{pos}(x, o) = \mathbf{p}\}$$

is open in G/H_1.

Lemma C.1.3 implies the following. Let o be a point in G/H_2 and H^o be its stabilizer in G. Then the union of open orbits of the action

$$\mathrm{H}^o \curvearrowright \mathrm{G}/\mathrm{H}_1$$

coincides with the set of elements $x \in \mathrm{G}/\mathrm{H}_1$ for which $\mathbf{pos}(x, o)$ is maximal.

Proof of Lemma C.1.3. Let $\mathbf{p}$ be a relative position and take $g \in \mathrm{G}$ such that

$$\mathbf{p} = \mathrm{H}_1 g \mathrm{H}_2.$$

It is not hard to see that $\mathbf{p}$ is maximal if and only if $\mathrm{H}_1 g \mathrm{H}_2$ is an open subset of G. Further, since the projection

$$\mathrm{G} \to \mathrm{G}/\mathrm{H}_1$$

is open, this is equivalent to the fact that the set

$$\{x \in \mathrm{G}/\mathrm{H}_1 : \quad \mathbf{pos}(x, \mathrm{H}_2) = \mathbf{p}\}$$

is open. The result now follows from equivariance.

□

C.1.3 Fat ideals

In this section we suppose further that the quotient space $\mathrm{H}_1\backslash\mathrm{G}/\mathrm{H}_1$ is finite[1], which gives us a notion of transversality between pairs of points in G/H_1. We can then introduce a symmetric relation $\leftrightarrow$ on $\mathrm{H}_1\backslash\mathrm{G}/\mathrm{H}_2$ as follows. Let $\mathbf{p}$ and $\mathbf{p}'$ be two elements in $\mathrm{H}_1\backslash\mathrm{G}/\mathrm{H}_2$ and write

$$\mathbf{p} \leftrightarrow \mathbf{p}'$$

if there exist transverse elements x and x' in G/H_1 and an element o in G/H_2 such that

$$\mathbf{pos}(x, o) = \mathbf{p} \text{ and } \mathbf{pos}(x', o) = \mathbf{p}'.$$

An *ideal* is a subset $\mathbf{I}$ of relative positions satisfying the following property: for every $\mathbf{p} \in \mathbf{I}$ and every $\mathbf{p}' \in \mathrm{H}_1\backslash\mathrm{G}/\mathrm{H}_2$ such that $\mathbf{p}' \leq \mathbf{p}$ one has

$$\mathbf{p}' \in \mathbf{I}.$$

It is said to be *fat* if for every $\mathbf{p} \notin \mathbf{I}$ there exists $\mathbf{p}' \in \mathbf{I}$ satisfying

[1]In the next chapter we will assume moreover that H_1 is a self opposite parabolic subgroup of G, so this condition is satisfied.

$$\mathbf{p} \leftrightarrow \mathbf{p}'.$$

Example C.1.4. Consider again Example C.1.2.

- When $H_1 = H_2 = P_{\mathfrak{a}^+}$, the minimal fat ideal (with respect to the inclusion) is given by $\mathbf{I} = \{\mathbf{d}, \mathbf{e}, \mathbf{f}\}$.
- When $H_1 = P_{\mathfrak{a}^+}$ and $H_2 = PSO(2,1)$, the minimal fat ideal (with respect to the inclusion) is given by the set consisting in the unique minimal position.

A different example is discussed in [63, Example 5.1.11].

◇

Remark C.1.5. The previous definition of fat ideal is inspired by the work of Kapovich-Leeb-Porti [30]. Indeed, suppose that H_1 and H_2 are parabolic subgroups of G, with H_1 being self opposite. It can be seen in this case that an ideal $\mathbf{I}$ of $H_1\backslash G/H_2$ is fat if and only if it is fat in the sense of [30]. For a proof of this fact see [63, Example 5.1.4].

◇

C.2 Domains of discontinuity for Anosov subgroups

We now prove our main result (Theorem C.2.3), associating to a fat ideal in $P\backslash G/H$ (where P is a self opposite parabolic subgroup of G) and a P-Anosov representation ρ a domain of discontinuity for ρ in G/H. Further, in Subsection C.2.3 we show that the union of non maximal relative positions is always a fat ideal, providing a unified framework for the sets $\mathbf{\Omega}_\rho$ considered in Parts II and III of this thesis.

C.2.1 Properly discontinuous actions

Let Ξ be a discrete group acting on a smooth manifold X by smooth diffeomorphisms. We say that an open subset Ω of X is a *domain of discontinuity* for this action if it is Ξ-invariant and for every compact set $C \subset \Omega$ one has

$$\#\{g \in \Xi : \quad g \cdot C \cap C \neq \emptyset\} < \infty.$$

In this case, the action $\Xi \curvearrowright \Omega$ is said to be *properly discontinuous*.

In the context of this thesis, the basic example of a properly discontinuous action is the following: let X_G be the Riemannian symmetric space of G. Since for any $\tau \in \mathsf{X}_G$ one has

$$\mathsf{X}_G \cong G/K^\tau$$

and the subgroup K^τ is compact, it is easy to see that X_G itself is a domain of discontinuity for any discrete subgroup $\Xi < \mathrm{G}$.

Before going into the construction of domains of discontinuity, let us state an equivalent formulation of this notion which will be useful. Let $\Omega \subset \mathsf{X}$ be an open Ξ-invariant subset. Two points o and o' in Ω are said to be *dynamically related* if there exist sequences $o_n \to o$ in Ω and $g_n \to \infty$ in Ξ such that

$$g_n \cdot o_n \to o'.$$

In this case one says that o and o' are dynamically related *via* the sequence $\{g_n\}_{n\geq 0}$. The proof of the following is easy.

Lemma C.2.1. *Let $\Omega \subset \mathsf{X}$ be an open Ξ-invariant subset. Then Ω is a domain of discontinuity for Ξ if and only if no two points in Ω are dynamically related.*

□

C.2.2 Statement and proof of the result

Suppose that G is a connected semisimple Lie group with finite center and no compact factors. Fix an element $\tau \in \mathsf{X}_\mathrm{G}$, a Cartan subspace $\mathfrak{a} \subset \mathfrak{p}^\tau$ and a closed Weyl chamber $\mathfrak{a}^+$ of the system $\Sigma(\mathfrak{g}, \mathfrak{a})$. Let P_θ be a self opposite parabolic subgroup of G associated to some non empty subset θ of simple roots. Fix a closed subgroup H of G for which the set $\mathrm{P}_\theta\backslash\mathrm{G}/\mathrm{H}$ is finite.

Let $\rho : \Gamma \to \mathrm{G}$ be a P_θ-Anosov representation with limit map

$$\xi = \xi_{\rho,\theta} : \partial_\infty\Gamma \to \mathsf{F}_\theta.$$

The following lemma is the key step in the proof of the main result of this appendix. It is inspired by [30, Proposition 6.2].

Lemma C.2.2. *Let $\{\gamma_n\}_{n\geq 0}$ be a sequence in Γ going to infinity and suppose that*

$$U_\theta^\tau(\rho\gamma_n) \to \xi_+ \text{ and } S_\theta^\tau(\rho\gamma_n) \to \xi_-$$

as $n \to \infty$, for some flags $\xi_\pm \in \xi(\partial_\infty\Gamma)$. Let o and o' be two points in G/H which are dynamically related via $\rho\gamma_n$ and $\mathbf{p} \in \mathrm{P}_\theta\backslash\mathrm{G}/\mathrm{H}$ be a relative position satisfying

$$\mathbf{p} \leftrightarrow \mathbf{pos}(\xi_-, o).$$

Then one has $\mathbf{pos}(\xi_+, o') \leq \mathbf{p}$.

Proof. Since $\mathbf{p} \leftrightarrow \mathbf{pos}(\xi_-, o)$ we can find two transverse flags ξ_1 and ξ_2 in F_θ and a point $\tilde{o} \in \mathrm{G}/\mathrm{H}$ such that

$$\mathbf{pos}(\xi_1, \tilde{o}) = \mathbf{p} \text{ and } \mathbf{pos}(\xi_2, \tilde{o}) = \mathbf{pos}(\xi_-, o).$$

By acting on the triple $(\xi_1, \xi_2, \tilde{o})$ with an suitable element of G we may further suppose $\tilde{o} = o$. Since the action of G in a fixed relative postition is transitive (Remark C.1.1) we find an element $g \in \mathrm{G}$ such that

$$g \cdot (\xi_-, o) = (\xi_2, o).$$

The element $\xi' := g^{-1} \cdot \xi_1$ is thus transverse to ξ_- and satisfies

$$\mathbf{pos}(\xi', o) = \mathbf{p}.$$

Now take a sequence $o_n \to o$ in G/H such that $\rho\gamma_n \cdot o_n \to o'$ and write $o_n = g_n \cdot o$ for some sequence $g_n \to 1$ in G. For every n large enough, the flag $g_n \cdot \xi'$ is transverse to ξ_- and therefore Remark 1.5.2 together with condition (2.1.1) gives

$$(\rho\gamma_n)g_n \cdot \xi' \to \xi_+$$

as $n \to \infty$. Hence

$$\mathbf{pos}(\xi_+, o') \leq \mathbf{pos}((\rho\gamma_n)g_n \cdot \xi', (\rho\gamma_n)g_n \cdot o) = \mathbf{pos}(\xi', o) = \mathbf{p}.$$

□

Given an ideal $\mathbf{I}$ in $\mathrm{P}_\theta \backslash \mathrm{G}/\mathrm{H}$ we define

$$\mathbf{\Omega}^{\mathbf{I}}_\rho := (\mathrm{G}/\mathrm{H}) \setminus \bigcup_{x \in \partial_\infty \Gamma} \{o \in \mathrm{G}/\mathrm{H} : \quad \mathbf{pos}(\xi(x), o) \in \mathbf{I}\}. \tag{C.2.1}$$

One can see that $\mathbf{\Omega}^{\mathbf{I}}_\rho$ is Γ-invariant and open.

Now we can prove the main result of this part.

Theorem C.2.3. *Let* $\mathbf{I}$ *be a fat ideal in* $\mathrm{P}_\theta \backslash \mathrm{G}/\mathrm{H}$. *Then* $\mathbf{\Omega}^{\mathbf{I}}_\rho$ *is a domain of discontinuity for the action of* Γ *on* G/H *via* ρ.

Proof. Suppose by contradiction that o and o' are two points in $\mathbf{\Omega}^{\mathbf{I}}_\rho$ which are dynamically related via $\rho\gamma_n$, for some sequence $\gamma_n \to \infty$ in Γ. We may assume that

$$U^\tau_\theta(\rho\gamma_n) \to \xi_+ \text{ and } S^\tau_\theta(\rho\gamma_n) \to \xi_-$$

as $n \to \infty$, for some flags $\xi_\pm \in \xi(\partial_\infty \Gamma)$.

Since $\mathbf{pos}(\xi_-, o) \notin \mathbf{I}$ and $\mathbf{I}$ is fat, we find an element $\mathbf{p} \in \mathbf{I}$ such that

$$\mathbf{p} \leftrightarrow \mathbf{pos}(\xi_-, o).$$

By Lemma C.2.2 we have

$$\mathbf{pos}(\xi_+, o') \leq \mathbf{p}$$

and therefore $\mathbf{pos}(\xi_+, o')$ belongs to $\mathbf{I}$. Since $o' \in \mathbf{\Omega}_\rho^{\mathbf{I}}$, this yields the desired contradiction.

□

C.2.3 Existence of fat ideals

Without any further assumption the set $\mathbf{\Omega}_\rho^{\mathbf{I}}$ could be empty for a given P_θ-Anosov representation ρ and a fat ideal $\mathbf{I}$ in $\mathrm{P}_\theta\backslash\mathrm{G}/\mathrm{H}$.

We now show the that set $\mathbf{I}_0$ of non maximal positions is always a fat ideal. Thanks to Lemma C.1.3, Schottky type constructions provide examples of Anosov representations for which $\mathbf{\Omega}_\rho^{\mathbf{I}_0}$ is non empty (c.f. Example 2.1.6). Also, if $\mathbf{\Omega}_\rho^{\mathbf{I}_0}$ is non empty for some particular representation ρ, then for every small enough deformation ρ' of ρ, the set $\mathbf{\Omega}_{\rho'}^{\mathbf{I}_0}$ will be non empty (c.f. Proposition 2.1.2).

Note that it is implicit in the definition of fat ideal that the set of relative positions must contain at least two elements.

Lemma C.2.4. *Assume that* $\mathrm{P}_\theta\backslash\mathrm{G}/\mathrm{H}$ *contains at least two elements. If* $\mathbf{p} \in \mathrm{P}_\theta\backslash\mathrm{G}/\mathrm{H}$ *is maximal, then there exists a non maximal position* $\mathbf{p}' \in \mathrm{P}_\theta\backslash\mathrm{G}/\mathrm{H}$ *for which*

$$\mathbf{p} \leftrightarrow \mathbf{p}'.$$

Lemma C.2.4 has the following straightforward consequence.

Corollary C.2.5. *Assume that* $\mathrm{P}_\theta\backslash\mathrm{G}/\mathrm{H}$ *contains at least two elements and define*

$$\mathbf{I}_0 := \{\mathbf{p} \in \mathrm{P}_\theta\backslash\mathrm{G}/\mathrm{H} : \quad \mathbf{p} \text{ is not maximal}\}.$$

Then $\mathbf{I}_0$ *is a fat ideal.*

□

Remark C.2.6. The subsets $\mathbf{\Omega}_\rho$ considered in Parts II and III of this thesis coincide with $\mathbf{\Omega}_\rho^{\mathbf{I}_0}$ in each context.

◇

Proof of Lemma C.2.4. Note that since G is connected and $\mathrm{P}_\theta\backslash\mathrm{G}/\mathrm{H}$ contains at least two elements, non maximal positions do exist (c.f. Lemma C.1.3). Suppose by contradiction that every non maximal position $\mathbf{p}'$ is not related to $\mathbf{p}$ under $\leftrightarrow$, and fix a flag $\xi \in \mathsf{F}_\theta$ and a point $o \in \mathrm{G}/\mathrm{H}$ for which

$$\mathbf{pos}(\xi, o)$$

is not maximal. Since $\mathbf{p}$ is not related to $\mathbf{pos}(\xi, o)$ under $\leftrightarrow$, we conclude that for every flag ξ' transverse to ξ one has

$$\mathbf{pos}(\xi', o) \neq \mathbf{p}.$$

Since the set of flags in F_θ which are transverse to ξ is dense in F_θ, we obtain a contradiction with Lemma C.1.3. □

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